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Preface

This book is intended to aid the reader in passing the preliminary examinations in the Physics department in the University of California at Berkeley. In my view, the most effective method of study for these exams, and indeed for most physics exams, is to work through as many problems as possible. I expect that most of the material is familiar to the reader and thus this text will not read like a textbook. Quick summaries of important concepts are collected at the beginning of each section for the reader’s convenience, but by no means do I claim these to be complete.

The growth of this book was highly nonlinear. Problems accreted on the text as I discovered more and more useful and interesting ones. I have tried to organize the problems into an essentially coherent lot with a logical flow, but this effort is almost surely doomed. As mentioned, it is not meant to be a textbook, but an exercise book. It is intended to be read alongside appropriate undergraduate physics texts.

I have not followed a particular text for the Classical Mechanics section. However, in reviewing the material myself, I often looked to Morin’s wonderful text. I have tried to follow Griffiths’ texts for Electricity and Magnetism and for Quantum Mechanics, and Kittel & Kroemer for Statistical Mechanics. Thus, the chapter titles of these sections are the appropriate and corresponding chapter titles in those books. I hope that this will prove convenient for study purposes.

I wrote this book in the Summer of 2010 while teaching a review course for continuing and incoming graduate students at U.C. Berkeley. These students are mentioned in the text for problems whose solutions they presented in class. However, I would like to take a moment to thank them all here. Thanks to James Arrenmann, Aaron Bradley, Victoria Martin, Nesty Torres-Chicon, and Derek Vigil-Currey for their participation in this enjoyable experience and for their help in correcting many of the errors in the text. I would also like to thank Kacey Meaker and Patrick Varilly whose notes, solutions and advice proved invaluable to my preparation for the course. Finally, I would like to thank Anne Takizawa for asking me to do this in the first place, and for all her support.

Inevitably, many errors remain in this book. I implore the reader to report such errors, whether minute or catastrophic, to me via my email: kevinqg@yahoo.com. Thank you very much and happy problem solving!

Kevin T. Grosvenor
CONTENTS
Part I

Classical Mechanics
Chapter 1

Lagrangian Mechanics

The Lagrangian is \( L = T - V \), where each of \( L, T \) and \( V \) is generically a function of some number of positions, their time derivatives, and time: \( L(q_i, \dot{q}_i, t) \). Most often, however, they are not explicit functions of time since that generically breaks time translation symmetry and energy conservation along with it. The action is \( S = \int L \, dt \) and the principle of stationary action gives the Euler-Lagrange (E-L) equations of motion (EoM)

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0.
\]

(1.0.1)

This only works for conservative forces, of course, since these are the only kind that can be written as the gradient of a potential. We won’t consider nonconservative forces when doing Lagrangian mechanics (LM).

To pass to the Hamiltonian, define the canonical conjugate momenta

\[
p_i = \frac{\partial L}{\partial \dot{q}_i},
\]

(1.0.2)

and write the following purely in terms of \( p_i \) and \( q_i \):

\[
H = p_i \dot{q}_i - L.
\]

(1.0.3)

If \( T \) is homogeneous and quadratic in the \( \dot{q}_i \)'s and \( V \) is independent of the \( \dot{q}_i \)'s, then \( H = T + V \). The Hamilton-Jacobi (H-J) EoM are then

\[
\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}.
\]

(1.0.4)

Often, you will be asked to find the frequency of small oscillations (FoSO) around equilibrium for some particular system. This topic can get fairly complicated especially if there are numerous equilibria. However, the basic idea is simple: define equilibrium to be the configuration that accomplishes \( \dot{q}_i \big|_{q_i,eq} = 0 \) and \( \ddot{q}_i \big|_{q_i,eq} = 0 \), then expand the E-L EoM around \( q_{i,eq} \). The resulting equation ought to be simple harmonic for the deviation from equilibrium.
When faced with a LM problem that might also ask for the FoSO about equilibrium, your basic strategy should be

1. There may be several solutions to the equation defining equilibrium. Some of those solutions may be unstable. You should be able to see this when you expand around that point: the resulting equation will not describe simple harmonic motion.

2. Find a reduced set of independent generalized coordinates, $q_i$. Then, write the positions of the appropriate moving objects in terms of these.

3. Write the Lagrangian $L(q_i, \dot{q}_i, t)$.

4. Write and/or solve the Euler-Lagrange equations of motion.

5. Plug in $\dot{q}_i|_{q_i,eq} = 0$ and $\ddot{q}_i|_{q_i,eq} = 0$ into the EoM and solve for $q_i,eq$.

6. Set $q_i = q_i,eq + \epsilon_i$ and expand the EoM to lowest nontrivial order in $\epsilon_i$. If the motion is really oscillatory, then it should look like $\dddot{\epsilon}_i + \omega_i^2 \epsilon_i = 0$.

Things to pay attention to or watch out for:

1. Sometimes, there will be several possible equilibria. Some of those may be unstable. It may be obvious, intuitively, which equilibria are stable and unstable, but if you can’t tell easily, just expand the EoM around each one. The stable ones will yield simple harmonic oscillation equations whereas the unstable ones will generically yield exponential growth equations. See Problem 1.6 and 1.7.

2. Sometimes, you will be asked to find conserved quantities, or express things in terms of conserved quantities. Usually, the energy (or Hamiltonian) will automatically be a conserved quantity. If the Lagrangian has no explicit time dependence, then $H$ will be conserved. In addition, if $L$ is independent of some generalized coordinate, $q$, then the momentum conjugate to $q$, which is $\partial L/\partial \dot{q}$, will automatically be conserved. See Problem 1.3.

3. You need to make sure that your generalized coordinates are well defined in the vicinity of the equilibrium point, or else expanding there will give you nonsense. A simple change of definition of coordinates will usually suffice to cure this problem. See Problem 1.3.

4. Sometimes, you will need to impose constraints on various coordinates. This requires the machinery of Lagrange multipliers. See Problem 1.11 and 1.12.
1.1 Pendulum Attached to the top of a Disk

[Spring 2010 Classical (Afternoon), Problem 3] A pendulum is constructed by attaching a mass \( m \) to an extensionless string of length \( \ell \). The upper end of the string is connected to the uppermost point on a vertical disk of radius \( R \) \((R < \ell / \pi)\) as in the figure. Obtain the pendulum equation of motion and find the frequency of small oscillations. Find the line about which the angular motion extends equally in either direction (i.e. \( \theta_1 = \theta_2 \)).

![Figure 1.1: Spring 2010 Classical (Afternoon), Problem 3](image)

SOLUTION:

Step 1: From the diagram, we will write down the coordinates of the mass. First, let us find the coordinates of the point where the string loses contact with the disk:

\[
(x_c, y_c) = R (\cos \theta, \sin \theta). \tag{1.1.1}
\]

The length of the string in contact with the disk is \( \ell = \ell - R \left( \frac{\theta}{2} - \theta \right) \). Therefore, the length of the remainder of the string is \( \ell = \ell - R \left( \frac{\theta}{2} - \theta \right) \). The coordinates of the mass are

\[
(x, y) = (R \cos \theta + \ell \sin \theta, R \sin \theta - \ell \cos \theta). \tag{1.1.2}
\]

Step 2: Differentiating \((x, y)\) with respect to time yields

\[
(\dot{x}, \dot{y}) = (-R \sin \theta \dot{\theta} + \ell \cos \theta \dot{\theta} + \ell \sin \theta, R \cos \theta \dot{\theta} + \ell \sin \theta \dot{\theta} - \ell \cos \theta) = \ell \dot{\theta} (\cos \theta, \sin \theta), \tag{1.1.3}
\]

where use was made of \( \dot{\ell} = R \dot{\theta} \). The Lagrangian is

\[
L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - mg y = \frac{1}{2} m \dot{\theta}^2 - m g (R \sin \theta - \ell \cos \theta). \tag{1.1.4}
\]

Step 3: The E-L EoM reads

\[
0 = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} \]

\[
= \frac{d}{d\theta} (m \ddot{\theta} \ell^2) - m \ell \dot{\ell} \dot{\theta}^2 + m g R \cos \theta + m g \ell \sin \theta - m g \ell \cos \theta
\]

\[
= m \ddot{\theta} \ell^2 - 2 m \ell \dot{\ell} \dot{\theta}^2 + m R \ell \dot{\theta}^2 + m g \ell \sin \theta
\]

\[
= m \ddot{\theta} \ell^2 + m R \dot{\theta}^2 + m g \ell \sin \theta, \tag{1.1.5}
\]

where use was made of \( \dot{\ell} \equiv \partial \ell / \partial \theta = R \) and \( \dot{\ell} = R \dot{\theta} \), again.

We can write the pendulum equation of motion as

\[
\ddot{\theta} + \frac{R}{\ell - R \left( \frac{\theta}{2} - \theta \right)} \dot{\theta}^2 + \frac{g}{\ell - R \left( \frac{\theta}{2} - \theta \right)} \sin \theta = 0. \tag{1.1.6}
\]
Step 4: For the equilibrium angle, we set $\ddot{\theta} = 0$ and $\dot{\theta} = 0$ so that the EoM now reads

$$\frac{g}{\ell - R(\frac{\pi}{2} - \theta)} \sin \theta = 0 \implies \sin \theta = 0.$$  

(1.1.7)

This is solved by the equilibrium angle

$$\theta_{eq} = 0$$  

(1.1.8)

Step 5: Set $\theta = \theta_{eq} + \epsilon = \epsilon$ and expand the EoM to linear order in $\epsilon$:

$$\ddot{\epsilon} + \frac{g}{\ell - \frac{\pi}{2} R} \epsilon = 0 \implies \omega^2 = \frac{g}{\ell - \frac{\pi}{2} R}$$  

(1.1.9)

Note that $\omega^2$ is indeed positive since $R < \ell/\pi$. The line of equal amplitude oscillation is therefore vertical.

Aside: the FoSO is just the usual pendulum frequency for a pendulum of length $\ell - \frac{\pi}{2} R$. We could have guessed that from the beginning.
1.2 Hoop and Pulley

[A Morin 6.10] A mass \(M\) is attached to a massless hoop of radius \(R\) that lies in a vertical plane. The hoop is free to rotate about its fixed center. \(M\) is tied to a string which winds part way around the hoop, then rises vertically up and over a massless pulley. A mass \(m\) hangs on the other end of the string. Find the equation of motion for the angle of rotation of the hoop. What is the frequency of small oscillations? Assume that \(m\) moves only vertically, and assume \(M > m\).

\[\text{Figure 1.2: Hoop and Pulley}\]

**SOLUTION:**

Define the position of \(M\) relative to the center of the hoop: \((x, y)_M = R(\sin \theta, -\cos \theta)\). Define the position of \(m\) relative to where it would be if \(M\) were at the bottom of the hoop (i.e. \(\theta = 0\)): \(y_m = -R\theta\). Note: it is not a problem that we have different “origins” for the two masses since a change of origin can only change the potential by a constant. Set the gravitational potential of mass \(M\) to zero when \(\theta = \pi/2\) and for \(m\) when \(y_m = 0\) (i.e. \(\theta = 0\)). Then, the Lagrangian reads

\[L = \frac{1}{2}(M + m)R^2\dot{\theta}^2 + MgR\cos \theta + mgR\theta.\]  

(1.2.1)

The E-L EoM for \(\theta\) is

\[(M + m)R\ddot{\theta} = g(m - M \sin \theta).\]  

(1.2.2)

We set \(\ddot{\theta} = \dot{\theta} = 0\) and solve for the equilibrium angle:

\[\theta_{eq} = \sin^{-1}(m/M).\]  

(1.2.3)

It’s a good thing \(M > m\), or else there is no solution!

The equation for the deviation \(\epsilon = \theta - \theta_{eq}\) is

\[\ddot{\epsilon} + \left(\frac{Mg \cos \theta_{eq}}{(M + m)R}\right)\epsilon = 0.\]  

(1.2.4)

After writing \(\cos \theta_{eq} = \sqrt{1 - \sin^2 \theta_{eq}}\), we get

\[\omega = \left(\frac{M - m}{M + m}\right)^{1/4} \sqrt{\frac{g}{R}}\]  

(1.2.5)
1.3 Rolling in a Bowl

[Fall 2008 Classical (Morning), Problem 2] A particle of mass \( m \) is confined to move without friction on the surface of a sphere of radius \( R \). The position vector \( \mathbf{r} = (x, y, z) \) of the particle obeys \(|\mathbf{r}|^2 = x^2 + y^2 + z^2 = R^2\). The particle is in a gravitational field \( \mathbf{g} = (0, 0, -g) \).

(a) Find the Lagrangian.

(b) Calculate the energy.

(c) Write down the equations of motion and use them to determine \( d\theta/dt \) in terms of \( \theta \) and conserved quantities.

(d) Find the frequency of small oscillations about the bottom of the sphere.

**SOLUTION:**

(a) The Cartesian coordinates are given by

\[
x = R \sin \theta \cos \phi, \quad y = R \sin \theta \sin \phi, \quad z = R \cos \theta.
\] (1.3.1)

We need \(|\dot{\mathbf{r}}|^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2\). After a bit of algebra, we find the kinetic energy

\[
T = \frac{1}{2} m |\dot{\mathbf{r}}|^2 = \frac{1}{2} m R^2 (\dot{\phi}^2 + \dot{\theta}^2 \sin^2 \theta).
\] (1.3.2)

The potential is simply \( V = mgR \cos \theta \). Thus, the Lagrangian is

\[
L = T - V = \frac{1}{2} m R^2 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) - mgR \cos \theta.
\] (1.3.3)

(b) We need the Hamiltonian. The conjugate momenta are

\[
p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m R^2 \dot{\theta}, \quad p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m R^2 \dot{\phi} \sin^2 \theta.
\] (1.3.4)

The Hamiltonian (energy) is

\[
E = p_\theta \dot{\theta} + p_\phi \dot{\phi} - L = \frac{1}{2} m R^2 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) + mgR \cos \theta.
\] (1.3.5)

(c) We can write the Hamiltonian as

\[
E = \frac{1}{2} m R^2 \dot{\theta}^2 + \frac{p_\phi^2}{2 m R^2 \sin^2 \theta} + mgR \cos \theta.
\] (1.3.6)

Since \( L \) is \( \phi \)-independent and has no explicit time dependence, \( p_\phi \) and \( E \) are conserved quantities. Therefore, we can just solve for \( \dot{\theta} \) in the above equation:

\[
\dot{\theta} = \left( \frac{2E}{m R^2} - \frac{p_\phi^2}{m^2 R^4 \sin^2 \theta} - \frac{2g \cos \theta}{R} \right)^{1/2}.
\] (1.3.7)
This is a little bit tricky if you just try to apply the standard procedure without thinking carefully about what you’re doing. So, we should first intuit the answer: near the bottom, we should expect the mass to behave just like a pendulum of length $R$. Thus, we expect

$$\omega = \sqrt{g/R}.$$  \hspace{1cm} (1.3.8)

The tricky part has to do with the fact that the spherical coordinates are singular at the poles: $\phi$ is not well-defined there. Or, in other words, even small oscillations at the bottom may encompass all possible values for $\phi$. Thus, to do this correctly, we need to define coordinates that are not singular at the bottom. Alternatively, simply rotate the gravitational field to $\mathbf{g} = (g, 0, 0)$ so that $V = -mgR \sin \theta \cos \phi$ and the “bottom” of the sphere now corresponds to $\theta = \pi/2$ and $\phi = 0$. Now, the E-L EoM are

$$\ddot{\theta} = \frac{1}{2} \dot{\phi}^2 \sin 2\theta + \frac{g}{R} \cos \theta \cos \phi,$$
$$\ddot{\phi} + 2 \dot{\phi} \dot{\theta} \cot \theta = -\frac{g}{R} \csc \theta \sin \phi.$$  \hspace{1cm} (1.3.9)

Set $\theta = \frac{\pi}{2} + \delta$ and $\phi = 0 + \epsilon$ and expand to linear order in $\delta$, $\epsilon$ and their derivatives so that $\sin \theta = 1$, $\cos \theta = -\delta$, $\sin \phi = \epsilon$ and $\cos \phi = 1$. This yields the two oscillatory equations $\ddot{\delta} = -\frac{g}{R} \delta$ and $\ddot{\epsilon} = -\frac{g}{R} \epsilon$. Thus, $\omega = \sqrt{g/R}$. 
1.4 Disk within a Disk

[Fall 2009 Classical (Morning), Problem 1] A small disk of mass $M_d$ and moment of inertia $I$ rolls (without slipping) inside a larger hollow disk which is held fixed. The motion takes place in a plane, in the presence of a uniform gravitational field of acceleration $g$. The radius, $R$, of the smaller disk is half that of the larger disk. A small mass, $m_p$, is attached to the rim of the smaller disk so that it is at the center of the larger disk when the smaller disk is at its lowest point.

(a) Find the equations of motion for the center of the smaller disk in terms of $M_d$, $m_p$, $I$, $R$, and $g$.

(b) What is the frequency of small oscillations of the disk?

**SOLUTION:**

(a) In the diagram above, we have drawn the small disk having rolled slightly to the right by some rotation angle $\theta$ with respect to the center of the large disk. By the no-slip condition, the two red arclengths must be equal and so the angle subtended by $m_p$ relative to the center of the small disk is $2\theta$.

Let the origin of coordinates be the center of the large disk. The positions of the center of the small disk and the mass, $m_p$, are

$$\begin{align*}
(x_c, y_c) &= R(\sin \theta, -\cos \theta), \\
(x_m, y_m) &= 2R \sin \theta(1, 0).
\end{align*}$$

Curiously, the mass, $m_p$, just moves horizontally right and left! The Lagrangian is

$$L = \frac{1}{2} M_d(\dot{x}_c^2 + \dot{y}_c^2) + \frac{1}{2} I(2\dot{\theta})^2 + \frac{1}{2} m_p \dot{x}_m^2 + M_d g y_c$$
$$= (4I + M_d R^2 + 4m_p R^2 \cos^2 \theta) \dot{\theta}^2 + M_d g R \cos \theta.$$  \hspace{1cm} (1.4.2)

After some algebra, we find the E-L EoM:

$$\left(4I + (M_d + 4m_p \cos^2 \theta) R^2\right) \ddot{\theta} - 4m_p R^2 \dot{\theta}^2 \sin \theta \cos \theta + M_d g R \sin \theta = 0.$$  \hspace{1cm} (1.4.3)

(b) For small $\theta$, we have

$$\left(4I + (M_d + 4m_p) R^2\right) \ddot{\theta} + M_d g R \theta = 0,$$  \hspace{1cm} (1.4.4)

and, thus, the frequency of small oscillations is

$$\omega^2 = \frac{M_d g R}{4I + (M_d + 4m_p) R^2}.$$  \hspace{1cm} (1.4.5)
Aside: We could group $4m_p R^2$ with $4I$ instead: $4(I + m_p R^2)$. Then, we can interpret the $m_p R^2$ term as simply adding to the moment of inertia of the small disk, the moment of inertia of a point mass rotating around the center of the small disk.
1.5 Cube atop a Fixed Cylinder

[Spring 2007 Classical (Morning), Problem 1] A solid cube of uniform density and sides of $a$ is centered atop a horizontal fixed cylinder of radius $R$. The contact between the cube and the cylinder is perfectly rough.

(a) Show that the conditions for stable equilibrium of the slab, assuming no slipping, is $R > a/2$.

(b) Write the Lagrangian for small oscillations about the equilibrium.

(c) What is the frequency of small oscillations?

---

**SOLUTION:**

(a) Set up coordinates as in the diagram. The red segments are both of length $R\theta$ due to the no-slip condition.

![Figure 1.5: Spring 2007 Classical (Morning), Problem 1.](image)

The three dotted triangles are all similar. This allows us to determine the coordinates of the center of mass of the cube:

$$
(x, y) = (R \sin \theta - R\theta \cos \theta + \frac{a}{2} \sin \theta, R \cos \theta + R\theta \sin \theta + \frac{a}{2} \cos \theta).
$$

(b) The Lagrangian involves the kinetic and potential energies of the center of mass plus the rotational energy of the cube about an axis passing through the center of two opposite faces. Let us compute the moment of inertia for such rotation:

$$I = \frac{m}{a^3} \int_{-a/2}^{a/2} (x^2 + y^2) \, dx \, dy \, dz = \frac{2ma}{3a} x^3 \mid_{-a/2}^{a/2} = \frac{1}{6} ma^2.
$$
Thus, the Lagrangian is

\[
L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2 - mg
\]

\[
= \frac{1}{2}m \left[ (R\dot{\theta}\sin \theta + \frac{a}{2}\dot{\theta}\cos \theta)^2 + (R\dot{\theta}\cos \theta - \frac{a}{2}\dot{\theta}\sin \theta)^2 \right] + \frac{1}{12}ma^2\dot{\theta}^2
\]

\[
- mg(R\cos \theta + R\dot{\theta}\sin \theta + \frac{a}{2}\cos \theta)
\]

\[
= \frac{1}{2}mR^2\dot{\theta}^2 \left[ \theta^2 + \frac{5}{12} \left( \frac{a}{R} \right)^2 \right] - mR \left[ 1 + \frac{a}{2R} \right] \cos \theta + \theta \sin \theta].
\] (1.5.4)

For small oscillations, we must write the Lagrangian up to quadratic order in $\theta$ and its derivatives. Doing so just changes $\cos \theta$ to $1 - \frac{1}{2} \theta^2$ and $\sin \theta$ to $\theta$. This gives

\[
L \approx \frac{5}{24}ma^2\dot{\theta}^2 - \frac{1}{2}mg(R - \frac{a}{2})\dot{\theta}^2 - mg(R + \frac{a}{2}).
\] (1.5.5)

(c) The E-L EoM for $\theta$ is

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = \frac{5}{12}ma^2\ddot{\theta} + mg(R - \frac{a}{2})\theta = 0.
\] (1.5.6)

Thus, the frequency of small oscillations is

\[
\omega = \sqrt{\frac{12g(R - \frac{a}{2})}{5a^2}}.
\] (1.5.7)

We could put this into the form

\[
\omega = \sqrt{\frac{3}{5}} \sqrt{\frac{R^2 - (a/2)^2}{(a/2)^2}} \sqrt{\frac{g}{R + \frac{a}{2}}},
\] (1.5.8)

which is a geometric factor times the frequency of a length $R + \frac{a}{2}$ pendulum.
1.6 Bead on a Horizontal Revolving Hoop

[Morin 6.12] A bead is free to slide along a frictionless hoop of radius $r$. The plane of the hoop is horizontal, and the center of the hoop travels in a horizontal circle of radius $R$, with constant angular speed $\omega$, about a given point. Find the equation of motion for the angle $\theta$ shown. Also, find the frequency of small oscillations about the equilibrium point.

![Figure 1.7: Bead on a rotating hoop.](image)

**SOLUTION:**

**Step 1:** Let the origin be the pivot of the length $R$ stick connected to the hoop. Then, the position of $m$ is

$$ (x, y) = (R \cos \omega t + r \cos(\omega t + \theta), \ R \sin \omega t + r \sin(\omega t + \theta)) \quad (1.6.1) $$

**Step 2:** The velocity is

$$ (\dot{x}, \dot{y}) = (-R \omega \sin \omega t - r(\omega + \dot{\theta}) \sin(\omega t + \theta), \ R \omega \cos \omega t + r(\omega + \dot{\theta}) \cos(\omega t + \theta)) \quad (1.6.2) $$

After a bit of algebra, we find the Lagrangian

$$ L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) = m \left[ R^2 \omega^2 + r^2 (\omega + \dot{\theta})^2 + 2 R r \omega (\omega + \dot{\theta}) \cos \theta \right] \quad (1.6.3) $$

**Step 3:** The E-L EoM for $\theta$ reads

$$ r \ddot{\theta} + R \omega^2 \sin \theta = 0 \quad (1.6.4) $$

**Step 4:** Setting $\dot{\theta} = \dot{\theta} = 0$ gives $\theta_{eq} = 0$ (c.f. Aside 2 below.)

**Step 5:** Set $\theta = \epsilon$ and expand (1.6.4) to linear order:

$$ \ddot{\epsilon} + (R/r) \omega^2 \epsilon = 0 \quad \Rightarrow \quad \Omega = \omega \sqrt{R/r} \quad (1.6.5) $$
Aside 1: You should check that the limiting cases make sense. For \( R >> r \), you can solve the problem by saying that \( m \) lives in an effective gravity \( g = (R + r)\omega^2 \) and dropping the \( r \) since \( r << R \). Then this is just like a pendulum of length \( r \) in an effective gravity \( g = R\omega^2 \). The frequency is \( \sqrt{g/r} = \omega\sqrt{R/r} \), as we found in (1.6.5).

Aside 2: In Step 4, we said that the equilibrium angle was 0. But, all we know is that its sine vanishes; could it not be \( \theta_{eq} = \pi \)? The answer is that this is an unstable equilibrium. We can see this by expanding the EoM. Set \( \theta = \pi + \epsilon \). Then,

\[
\ddot{\epsilon} + \Omega^2 \left[ \sin \pi \cos \epsilon + \cos \pi \sin \epsilon \right] \approx \ddot{\epsilon} - \Omega^2 \epsilon = 0.
\]

(1.6.6)

The term proportional to \( \epsilon \) has the wrong sign. This is the equation for exponential growth rather than oscillatory motion. To see this, multiply by \( 2\dot{\epsilon} \):

\[
2\ddot{\epsilon} - \Omega^2 (2\dot{\epsilon}^2) = \frac{d}{d\epsilon} \left( \dot{\epsilon}^2 - \Omega^2 \epsilon^2 \right) = 0.
\]

(1.6.7)

Hence, up to a constant, which can be set to zero by appropriate initial conditions,

\[
\dot{\epsilon} = \Omega \epsilon,
\]

(1.6.8)

whose solution grows exponentially \( \epsilon \sim e^{\Omega t} \).
1.7 Bead on a Vertical Rotating Hoop

[Fall 2007 Classical (Afternoon), Problem 1; Jose & Saletan Ex. 2.2.1] A bead of mass \( m \) slides without friction in a uniform gravitational field on a vertical hoop of radius \( R \). The hoop is constrained to rotate at a fixed angular velocity about its vertical diameter. Let \( \theta \) be the angular position of the bead relative to the lowest point.

(a) Write down the Lagrangian \( L(\theta, \dot{\theta}) \).

(b) How do the equilibrium values of \( \theta \) depend on \( \Omega \). [Hint: There are three equilibrium points.] Which of these three points are stable and which are unstable?

(c) Find the frequencies of small vibrations about the stable equilibrium positions. Say something about the motion about the only stable equilibrium point when \( \Omega = \sqrt{g/R} \).

---

\[ L(\theta, \dot{\theta}) = T - V = \frac{1}{2} m R^2 (\dot{\theta}^2 + \Omega^2 \sin^2 \theta) + mg R \cos \theta. \tag{1.7.1} \]

(b) The E-L EoM for \( \theta \) is

\[ 0 = \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = m R^2 \ddot{\theta} - m R^2 \Omega^2 \sin \theta \cos \theta + mg R \sin \theta, \tag{1.7.2} \]

or, more neatly,

\[ \ddot{\theta} - \Omega^2 \sin \theta \cos \theta + \frac{g}{R} \sin \theta = 0. \tag{1.7.3} \]

At equilibrium, all time derivatives of \( \theta \) vanish. Then (1.7.3) reads

\[ 0 = -\Omega^2 \sin \theta \left( \cos \theta - \frac{g}{\Omega^2 R} \right). \tag{1.7.4} \]

There are three solutions within the range \( \theta \in [0, \pi] \):

\[ \theta_0 = \cos^{-1} \left( \frac{g}{\Omega^2 R} \right), \quad \theta_1 = 0, \quad \theta_2 = \pi. \tag{1.7.5} \]
1.7. BEAD ON A VERTICAL ROTATING HOOP

Write \( \theta = \theta_j + \epsilon_j \) as an expansion around each equilibrium point, for \( j = 0, 1, 2 \). Via the Taylor expansion, \( \sin \theta \approx \sin \theta_j + \epsilon_j \cos \theta_j \) and \( \cos \theta \approx \cos \theta_j - \epsilon_j \sin \theta_j \). Thus, to linear order in \( \epsilon_j \), (1.7.3) reads

\[
0 = \ddot{\epsilon}_j - \Omega^2 (\sin \theta_j + \epsilon_j \cos \theta_j) (\cos \theta_j - \epsilon_j \sin \theta_j) + \frac{g}{R} (\sin \theta_j + \epsilon_j \cos \theta_j)
\]

\[= \ddot{\epsilon}_j + \left[ \frac{g}{R} \cos \theta_j - \Omega^2 (\cos^2 \theta_j - \sin^2 \theta_j) \right] \epsilon_j + \frac{g}{R} \sin \theta_j - \Omega^2 \sin \theta_j \cos \theta_j \]

\[= \ddot{\epsilon}_j + \left[ \frac{g}{R} \cos \theta_j - \Omega^2 (2 \cos^2 \theta_j - 1) \right] \epsilon_j + \frac{g}{R} \sin \theta_j - \Omega^2 \sin \theta_j \cos \theta_j. \tag{1.7.6}
\]

\( j = 0 \): Then, \( \cos \theta_0 = \frac{g}{\Omega^2 R} \) and (1.7.6) reads

\[
0 = \ddot{\epsilon}_0 + \left\{ \left( \frac{g}{\Omega^2 R} \right)^2 - \Omega^2 \left[ 2 \left( \frac{g}{\Omega^2 R} \right)^2 - 1 \right] \right\} \epsilon_0 + \frac{g}{R} \sin \theta_0 - \Omega^2 \sin \theta_0 \left( \frac{2g}{\Omega^2 R} \right)
\]

\[= \ddot{\epsilon}_0 + \Omega^2 \left[ 1 - \left( \frac{2g}{\Omega^2 R} \right)^2 \right] \epsilon_0. \tag{1.7.7}
\]

This takes the form of simple harmonic motion if and only if the prefactor of \( \epsilon_0 \) is positive. Thus,

\[ \theta_0 \text{ is stable if and only if } \Omega > \sqrt{g/R}. \tag{1.7.8} \]

Note that \( 0 \leq \theta_0 \leq \pi/2 \).

\( j = 1 \): Then \( \sin \theta_1 = 0, \cos \theta_1 = 1 \) and (1.7.6) reads

\[
0 = \ddot{\epsilon}_1 + \left( \frac{g}{\Omega^2 R} - \Omega^2 \right) \epsilon_1
\]

\[= \ddot{\epsilon}_1 + \Omega^2 \left( \frac{g}{\Omega^2 R} - 1 \right) \epsilon_1. \tag{1.7.9}
\]

Requiring that the prefactor of \( \epsilon_1 \) be positive implies

\[ \theta_1 \text{ is stable if and only if } \Omega < \sqrt{g/R}. \tag{1.7.10} \]

\( j = 2 \): Then \( \sin \theta_2 = 0, \cos \theta_2 = -1 \) and (1.7.6) reads

\[
0 = \ddot{\epsilon}_2 + \left( -\frac{g}{\Omega^2 R} - \Omega^2 \right) \epsilon_2
\]

\[= \ddot{\epsilon}_2 - \Omega^2 \left( 1 + \frac{g}{\Omega^2 R} \right) \epsilon_2. \tag{1.7.11}
\]

The prefactor of \( \epsilon_2 \) cannot be positive and thus,

\[ \theta_2 \text{ is always unstable}. \tag{1.7.12} \]

\( \text{(c)} \) From (1.7.7) and (1.7.9), the frequencies are

\[
\omega = \Omega \times \begin{cases} (\frac{g}{\Omega^2 R} - 1)^{1/2}, & \text{if } \Omega < \sqrt{g/R}, \\ 1 - (\frac{g}{\Omega^2 R})^{1/2}, & \text{if } \Omega > \sqrt{g/R}. \end{cases}
\tag{1.7.13}
\]

If \( \Omega = \sqrt{g/R} \), then this is no longer harmonic; we must expand to \( O(\epsilon^2) \) or higher.
1.8 Pendulum Hanging from a Hoop

[Kevin G.] A pendulum of length $\ell$ and mass $m$ (concentrated at the end) hangs freely from the top of a vertical circular hoop of radius $R$ (with $2R > \ell$), whose mass you may neglect. The hoop is free to roll without slipping. Show that the system cannot execute simple harmonic oscillation unless the hoop is fixed in place. Otherwise, if $\omega_0$ is the initial angular speed of the hoop (and $\phi_0 = 0$ is the initial angular position), then solve for $\theta(t)$ in the regime when both $\phi(t)$ and $\theta(t)$ are small. Let the initial conditions for $\theta$ be $\theta_0 = 0$ and $\dot{\theta}_0 = 0$.

![Figure 1.9: Pendulum Hanging from a hoop.](image)

**SOLUTION:**

Since we will neglect the mass of the hoop, only $m$ contributes to $L$. Let the origin be the position of $m$ when the system is motionless and the pendulum just hangs vertically (left diagram.) The general position of $m$ is

$$(x, y) = (R\dot{\phi} + R\sin\phi + \ell\sin\theta, -R(1 - \cos\phi) + \ell(1 - \cos\theta)).$$  \hspace{1cm} (1.8.1)

The $R\dot{\phi}$ term in $x$ is the horizontal distance that the center of the hoop moves while rolling without slipping.

We take $m$ to have zero gravitational potential energy at our above choice of origin. After some algebra, we find the Lagrangian

$$L = mR^2\dot{\phi}^2(1 + \cos\phi) + mR\ell\dot{\phi}\dot{\theta}((\cos\phi + \cos\theta) + \frac{1}{2}m\ell^2\dot{\theta}^2 - m\ell(1 - \cos\phi) + mgR(1 - \cos\phi)).$$  \hspace{1cm} (1.8.2)

After some more mindless algebra, we find the E-L EoM

$$2R\ddot{\phi}(1 + \cos\phi) - R\dot{\phi}^2\sin\phi - g\sin\phi - \ell\dot{\theta}^2(\sin\phi + \sin\theta) = 0,$$  \hspace{1cm} (2.7.3a)

$$R\ddot{\phi}(\cos\phi + \cos\theta) - R\dot{\phi}^2\sin\phi + \ell\dot{\theta} + g\sin\theta = 0.$$  \hspace{1cm} (2.7.3b)

Expand both to linear order in $\phi$ and $\theta$:

$$4R\ddot{\phi} - g\phi = 0,$$  \hspace{1cm} (2.7.4a)

$$2R\ddot{\phi} + \ell\ddot{\theta} + g\theta = 0.$$  \hspace{1cm} (2.7.4b)

The term linear in $\phi$ in the first equation has the wrong sign to be oscillatory. Instead it is exponential (both growing and decaying), and thus unstable! Obviously, if $\phi$ is fixed, then we just recover the usual pendulum equation for $\theta$ with frequency $\omega = \sqrt{g/\ell}$. Otherwise, we set

$$\tau = \sqrt{4R/g}$$

and

$$\phi(t) = \omega_0\tau \sinh(t/\tau),$$  \hspace{1cm} (1.8.5)

where the coefficients of the exponentially growing and decaying parts were determined by the initial conditions, $\phi_0 = 0$ and $\dot{\phi}_0 = \omega_0$.

Then, the equation for $\theta$ now reads (where $\omega = \sqrt{g/\ell}$)

$$\ddot{\theta} + \omega^2\theta = -\frac{1}{2}\omega^2\omega_0\tau \sinh(t/\tau),$$  \hspace{1cm} (1.8.6)
which is an exponentially-forced harmonic oscillator.

First, we must solve the homogeneous (unforced) equation in general:

\[ \theta_{\text{gen}}(t) = A \sin \omega t + B \cos \omega t. \]  \hfill (1.8.7)

Then, we add to this a specific solution to the inhomogeneous (forced) equation. Since the right hand side of (1.8.6) contains a sinh, an obvious ansatz is

\[ \theta_{\text{spec}}(t) = C \sinh(t/\tau). \]  \hfill (1.8.8)

Plugging this in, solving for \( C \), and adding \( \theta_{\text{spec}} \) to \( \theta_{\text{gen}} \) gives

\[ \theta(t) = A \sin \omega t + B \cos \omega t - \frac{1}{2} \left( \frac{(\omega \tau)^2 \omega_0 \tau}{1 + (\omega \tau)^2} \right) \sinh(t/\tau). \]  \hfill (1.8.9)

The initial condition \( \theta_0 = 0 \) implies \( B = 0 \) and \( \dot{\theta}_0 = 0 \) gives

\[ A \left( \frac{\omega_0 \tau^2}{1 + (\omega \tau)^2} \right). \]  \hfill (1.8.10)

Thus, the solution is

\[ \theta(t) = \frac{1}{2} \left( \frac{\omega_0 \tau^2}{1 + (\omega \tau)^2} \right) (\sin \omega t - \omega \tau \sinh(t/\tau)). \]  \hfill (1.8.11)

In terms of the original variables,

\[ \theta(t) = \frac{2R \sqrt{l/g}}{l + 4R} \left( \sin(t \sqrt{g/l}) - \sqrt{4R/l} \sinh(t \sqrt{g/4R}) \right). \]  \hfill (1.8.12)
1.9 Swinging Masses on a Table

[Spring 201 Classical Mechanics, Problem 4] Consider two equal point masses connected by a string of length $L$ which passes through a hole on a table. The suspended mass moves up and down only. Ignore friction.

(a) Write down the Lagrangian of the system.

(b) Derive the equation of motion for $r$ and $\theta$.

(c) If the initial angular momentum around the center of the circle is $\ell$, show that there exists a solution of uniform circular motion with the radius $r_0 = (\ell^2/m^2 g)^{1/3}$.

(d) Show that for small deviations of $r$ around $r_0$, $r$ oscillates around $r_0$. Derive the oscillation frequency.

\[ \text{Figure 1.10: Spring 2011 Classical Mechanics, Problem 4.} \]

SOLUTION:

(a) Define potential to be zero at the level of the table and increase positively upwards. Define the $+z$ direction to be upwards. The Lagrangian is

\[ L = \frac{1}{2} m (\dot{r}^2 + \dot{\theta}^2) + \frac{1}{2} m \left[ \frac{d}{dt} (L - r) \right]^2 - mg[-(L - r)] \]
\[ = \frac{1}{2} m (2\dot{r}^2 + \dot{\theta}^2) + mg(L - r). \]  

(1.9.1)

(b) The $r$ and $\theta$ equations of motion are, respectively,

\[ 2m\ddot{r} - mr^2\dot{\theta}^2 + mg = 0, \quad mr^2\ddot{\theta} + 2m\dot{r}\dot{\theta} = 0. \]

(1.9.2)

The angular momentum is $\ell \equiv mr^2\dot{\theta}$. These equations may instead be written as

\[ m^2 r^3 (2\dddot{r} + g) - \ell^2 = 0, \quad \dot{\ell} = 0. \]  

(1.9.3)

(c) Circular motion implies setting $r \equiv r_0$ and the vanishing of all time derivatives of $r$. Thus, the first equation in (1.9.3) now reads $m^2 r_0^3 g - \ell^2 = 0$, which yields the solution $r_0 = (\ell^2/m^2 g)^{1/3}$.

(d) Plugging $r = r_0 (1 + \epsilon)$ into the same equation and writing $\ell^2$ in terms of $r_0$ gives the equation $m^2 r_0^3 (1 + 3\epsilon + 3\epsilon^2 + \epsilon^3)(2\dddot{\epsilon} + g) - m^2 r_0^3 g = 0$. Keeping only up to linear terms in $\epsilon$ gives the equation $\dddot{\epsilon} + \frac{3g}{2r_0} \epsilon = 0$ and thus, $\omega = \sqrt{3g/2r_0}$. 

\[ \text{Figure 1.10: Spring 2011 Classical Mechanics, Problem 4.} \]
1.10 Coupled Pendulum and Spring Cart

[Spring 2009 Classical (Morning), Problem 4; Taylor 7.31] A simple pendulum (mass \( M \) and length \( L \)) is suspended from a cart (mass \( m \)) that can oscillate on the end of a spring of force constant \( k \). Ignore friction.

(a) Find the equations that describe the motion in terms of the two coordinates \( x \) and \( \phi \), where \( x \) is the extension of the spring from its equilibrium length.

(b) Simplify the equations to the case when both \( x \) and \( \phi \) are small.

(c) Find an equation that defines the oscillation frequency(ies) in terms of \( m, M, L, k \) and \( g \). You need not solve this equation.

![Figure 1.11: Spring 2009 Classical (Morning), Problem 4.](image)

**SOLUTION:**

(a) **Step 1:** Define the origin to be the point where the pendulum is attached to the cart when the spring is at its equilibrium length. Define the horizontal and vertical components of the position of the pendulum:

\[
X = x + L \sin \phi, \quad Y = -L \cos \phi. \tag{1.10.1}
\]

**Step 2:** The total kinetic energy of the system is

\[
T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}M\dot{X}^2 + \frac{1}{2}MY^2
= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}M\left(\dot{x} + L\dot{\phi} \cos \phi\right)^2 + \frac{1}{2}ML^2\dot{\phi}^2 \sin \phi
= \frac{1}{2}(m + M)\dot{x}^2 + ML\dot{\phi} \cos \phi + \frac{1}{2}ML^2\dot{\phi}^2 \cos^2 \phi + \frac{1}{2}ML^2\dot{\phi}^2 \sin^2 \phi
= \frac{1}{2}(m + M)\dot{x}^2 + ML\dot{\phi} \cos \phi + \frac{1}{2}ML^2\dot{\phi}^2. \tag{1.10.2}
\]

The first term is the rigid translational kinetic energy of the whole system and the third term is the rotational kinetic energy of the pendulum by itself. The second term looks tantalizingly like a Coriolis-type term.

The potential energy for this system is

\[
V = \frac{1}{2}kx^2 + MgY = \frac{1}{2}kx^2 - MgL \cos \phi. \tag{1.10.3}
\]

Therefore, the Lagrangian is

\[
\mathcal{L} = T - V = \frac{1}{2}(m + M)\dot{x}^2 + ML\dot{\phi} \cos \phi + \frac{1}{2}ML^2\dot{\phi}^2 - \frac{1}{2}kx^2 + MgL \cos \phi. \tag{1.10.4}
\]

**Step 3:** Let us compute the following:

\[
\frac{\partial \mathcal{L}}{\partial x} = -kx,
\]

\[
\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{d}{dt} \left[ (m + M)\dot{x} + ML\dot{\phi} \cos \phi \right]
= (m + M)\ddot{x} + ML\ddot{\phi} \cos \phi - ML\dot{\phi}^2 \sin \phi. \tag{1.10.5}
\]
Therefore, the E-L EoM for $x$ is

$$\left( m + M \right) \ddot{x} + k x + M L \ddot{\phi} \cos \phi - M L \dot{\phi}^2 \sin \phi = 0.$$  \hfill (1.10.6)

Likewise, we compute

$$\frac{\partial L}{\partial \dot{\phi}} = -M L \dot{\phi} \sin \phi - M g L \sin \phi,$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\phi}} = \frac{d}{dt} \left[ M L \dot{x} \cos \phi + M L^2 \dot{\phi} \right] = M L \dot{x} \cos \phi - M L \dot{\phi} \sin \phi + M L^2 \ddot{\phi}.$$  \hfill (1.10.7)

Thus, the E-L EoM for $\phi$ is

$$\ddot{x} \cos \phi + L \ddot{\phi} + g \sin \phi = 0.$$  \hfill (1.10.8)

(b) Up to linear terms in $x, \dot{x}, \phi$ and $\dot{\phi}$, all assumed small, the EoM simplify to

$$\left( m + M \right) \ddot{x} + k x + M L \ddot{\phi} = 0,$$

$$\ddot{x} + L \ddot{\phi} + g \phi = 0.$$  \hfill (1.10.9)

(c) Consider the normal mode (equal frequency) ansatz

$$x = Ae^{i\omega t}, \quad \phi = Be^{i\omega t}.$$  \hfill (1.10.10)

Plugging these into (1.10.9) yields the following conditions:

$$\left[ -(m + M) \omega^2 + k \right] A - M L \omega^2 B = 0, \quad -\omega^2 A + (g - L \omega^2) B = 0.$$  \hfill (1.10.11)

After some algebra, these yield a quartic equation for $\omega$:

$$\omega^4 - \left[ (m + M) g + k L \right] \omega^2 + \frac{kg}{mL} = 0.$$  \hfill (1.10.12)

For convenience, define the following frequencies and mass ratio:

$$\Omega_s^2 = \frac{k}{m}, \quad \Omega_p^2 = \frac{g}{L}, \quad \mu = \frac{M}{m}.$$  \hfill (1.10.13)

$\Omega_s$ would be the oscillation frequency of the cart and spring without the pendulum and $\Omega_p$ that of the pendulum by itself. Then, (1.10.12) may be written

$$\omega^4 - \left[ (1 + \mu) \Omega_p^2 + \Omega_s^2 \right] \omega^2 + \left( \Omega_p \Omega_s \right)^2 = 0.$$  \hfill (1.10.14)

Aside: Let us compute the discriminant, $\mathcal{D}$, of the quadratic equation:

$$\mathcal{D} = \left[ (1 + \mu) \Omega_p^2 + \Omega_s^2 \right]^2 - 4 \left( \Omega_p \Omega_s \right)^2 = \left[ (\Omega_p + \Omega_s)^2 + \mu \Omega_p^2 \right] \left[ (\Omega_p - \Omega_s)^2 + \mu \Omega_p^2 \right].$$  \hfill (1.10.15)

This has the fortunate property of being positive definite! Therefore, there will always be two real solutions and since $\sqrt{\mathcal{D}}$ is smaller than negative of the $\omega^2$ coefficient, both solutions will always be positive. Since there are two components, there had better generically be two independent normal modes!
Let’s consider some limiting cases. Suppose that $\mu << 1$, or that the cart is much heavier than the pendulum. In this case, the two solutions for $\omega$ are approximately $\omega = \Omega_s$ and $\omega = \Omega_p$. The mode $\omega = \Omega_s$ is described by the heavy cart oscillating more or less as if the very light pendulum were not even there and the pendulum just goes along for the ride. Interestingly, if $\Omega_s < \Omega_p$, then $A$ and $B$ have the same sign, as most easily seen from the second equation in (1.10.11); that is, the cart and pendulum move in the same direction. If $\Omega_s > \Omega_p$, then $A$ and $B$ have opposite signs and the pendulum and cart move in opposite directions. The mode $\omega = \Omega_p$ is the case when the cart is hardly moving ($A \to 0$) and the pendulum swings as usual.

The $\mu >> 1$ limit is more complicated: to get the lower frequency, one must expand in $1/\mu$ up to quadratic order.
1.11 Rolling off a Bump

[Spring 2008 Classical (Afternoon), Problem 1; Thornton & Marion 7.10] A particle of mass $M$ starts at rest on top of a smooth (frictionless) hemisphere of radius $a$. The particle initially slides down the hemispherical surface, but at some point it loses contact with the surface and starts falling freely through the air. Find the angle $\theta$ (from the vertical) at which it loses contact with the surface.

SOLUTION:

Method 1: This problem is really quite easy to solve using Newtonian mechanics; LM is unnecessarily complicated in this case. So, let’s solve it the easy way first.

While the particle is in contact with the hemisphere, the forces on it are: (1) gravity, $F_g = -Mg\hat{y} = -Mg \cos \theta \hat{r} + Mgsin \theta \hat{\theta}$; and (2) normal force, $F_n = N \hat{r}$. The net force towards the center, which is therefore the centripetal force, is

$$F_c = -F_c \hat{r} \text{ where } F_c = Mg \cos \theta - N.$$  

But, we also have $F_c = Mv^2/a$, where $v$ is the tangential speed of the particle, which we can determine by energy conservation. At $\theta$, the vertical distance travelled by the particle is $a(1 - \cos \theta)$ and so the gravitational potential energy lost, or kinetic energy gained, is $Mga(1 - \cos \theta)$, which is equal to $\frac{1}{2}Mv^2$. Thus,

$$F_c = Mv^2/a = 2Mg(1 - \cos \theta).$$  

Combining the above two equations and dividing by $2Mg$ yields

$$1 - \cos \theta = \frac{1}{2} \cos \theta - \frac{N}{2Mg}.$$  

Right when the particle loses contact, $N = 0$ (by definition, in fact). Then we get

$$\theta = \cos^{-1}(2/3).$$  

Method 2: Now, we will use Lagrangians. Ideally, if we were to draw the confining potential on the particle just due to the hemisphere (not including gravity), it would be an infinite step function in $r$: it vanishes for $r > a$ and is $+\infty$ for $r < a$. But, that’s not so realistic: the hemisphere will have some finite elasticity and the ball will sink ever so slightly at the surface. Model this using a smooth potential, $V(r)$, that vanishes for $r > a$ and rises very steeply over a very short distance just below $r = a$.

Then, the Lagrangian reads

$$L = \frac{1}{2}M(\dot{r}^2 + r^2 \dot{\theta}^2) - Mgr \cos \theta - V(r).$$  

You can work out the E-L EoM to be

$$\ddot{\theta} = \frac{g}{r} \sin \theta - \frac{2\dot{r}\dot{\theta}}{r},$$  

$$\dot{\theta}^2 = \frac{g}{r} \cos \theta + \frac{\dot{r}^2}{r} + \frac{V'(r)}{Mr}.$$  

1.11. **ROLLING OFF A BUMP**

While the particle is in contact, \( r = a \) is fixed, and the EoM simplify to

\[
\ddot{\theta} = \frac{g}{a} \sin \theta, \tag{1.10.7a}
\]

\[
\dot{\theta}^2 = \frac{g}{a} \cos \theta + \frac{V'(a)}{Ma}. \tag{1.10.7b}
\]

Multiply Eqn. (1.10.7a) by \( 2\dot{\theta} \) and integrate. The initial condition \( \dot{\theta} = 0 \) when \( \theta = 0 \) gives

\[
\dot{\theta}^2 = \frac{2g}{a} (1 - \cos \theta). \tag{1.11.8}
\]

Now, since \( V(r) \) is smooth and vanishes for \( r > a \), we must have \( V'(a) = 0 \). Thus,

\[
\dot{\theta}^2 = \frac{g}{a} \cos \theta. \tag{1.11.9}
\]

Combining the last two equations yields the same result, Eqn. (1.11.4).

**Aside:** What I have done here, by introducing \( V(r) \), is, in my mind, a more physically motivated version of Lagrange multipliers. Our constraint is \( g = r - a \). We simply add a term \(-\lambda \frac{\partial L}{\partial \dot{r}} = -\lambda \) to the E-L EoM for \( r \) of the standard Lagrangian, Eqn. (1.11.5), without the \( V(r) \) term. Comparing with Eqn. (1.10.6b), you should see that \( \lambda \) is just \(-V'(r)\), which is precisely the constraint force. We set this equal to zero right when the ball loses contact with the hemisphere.
1.12 Yo-Yo

[Taylor 7.14] Model a yo-yo by a string wrapped once around the perimeter of a disk of radius $R$ and mass $M$. Suppose, however unrealistic it may be, that the string unrolls as in the diagram, that is the free string remains vertical while the disk falls and rolls without slipping.

(a) Write down the Lagrangian for the yo-yo as a function solely of the vertical distance that the yo-yo has fallen and determine the E-L EoM.

(b) Now, solve this problem again by introducing the appropriate constraint and Lagrange multiplier. Solve for the multiplier and interpret it physically (i.e. what does it represent?)

Figure 1.13: Falling yo-yo.

SOLUTION:

(a) The moment of inertia of a solid disk around its center is $I = \frac{1}{2}MR^2$. Let the origin of coordinates be the tip of the string right at the beginning when it is completely wrapped around the disk. Let $z$ be the vertical position of the yo-yo after falling a distance $z$. Define positive angle for the rotation of the yo-yo to be counter-clockwise. Rolling without slipping implies $z = -R\theta$ (do you understand the minus sign?) and that the rotation frequency is $\omega = \dot{\theta} = -\dot{z}/R$. Therefore, the Lagrangian is

$$L = \frac{1}{2}M\ddot{z}^2 + \frac{1}{2}I\Omega^2 - Mg(-z) = \frac{3}{4}M\ddot{z}^2 + Mgz.$$  \hspace{1cm} (1.12.1)

The E-L EoM simply reads $\ddot{z} = \frac{2g}{3}$.

(b) The constraint is $g(z, \theta) = z - R\theta$ (i.e. $g(z, \theta) = 0$). Thus, the Lagrangian is

$$L = \frac{1}{2}M\ddot{z}^2 + \frac{1}{2}I\dot{\theta}^2 + Mgz + \lambda(z - R\theta).$$  \hspace{1cm} (1.12.2)

The $z$ and $\theta$ E-L EoM read

$$M\ddot{z} = Mg + \lambda, \quad I\ddot{\theta} = -\lambda R.$$  \hspace{1cm} (1.12.3)

The last equation is the constraint itself, which gives us three equations for the three unknowns: $z$, $\theta$ and $\lambda$. The constraint equation implies $\ddot{z} = R\ddot{\theta}$, which, when combined with (1.12.3) gives

$$g + \frac{\lambda}{M} = -\frac{R^2\lambda}{I}.$$  \hspace{1cm} (1.12.4)

Finally, plugging in $I = \frac{1}{2}MR^2$ gives $\lambda = -mg/3$. Plugging this into the EoM gives the same EoM as in part (a). From $\ddot{z} = 2g/3$, it is clear that the tension force on the string must be $-mg/3$. Thus, the Lagrange multiplier represents the tension force in the string.
Chapter 2

More Mechanics

This chapter will be devoted to more problems in Mechanics that do not necessarily or directly involve Lagrangians.

Problems and topics:

1. Balancing forces: 2.1, 2.2, 2.3.
2. Normal modes: 1.10, 2.4, 2.5.
4. Relativity and relativistic collisions: 2.7, 2.8, 2.9, 2.10, 2.11.
6. Fictitious forces: 2.15, 2.16, 2.17.
7. Bernoulli’s equation: 2.17, 2.18.
10. Air Resistance: 2.21.

Quick summaries:

1. Balancing forces:
   - \( \mathbf{F} = m \mathbf{a} \) for non-changing mass and non-relativistic cases, and \( \mathbf{F} = d \mathbf{p}/dt \) otherwise.
   - Rotation: \( \mathbf{F}_c = m v^2/r \).
   - Moment of inertia: \( I = \int r^2 \, dm \). You may want to memorize some common ones. Personally, I always rederive \( I \) since I’ve done it enough times so that I can do it relatively quickly.
   - Torque and angular momentum: \( \mathbf{L} = I \omega \) and \( \tau = \dot{\mathbf{L}} = I \alpha \).
   - Friction: \( F_f \leq \mu N \), where \( N \) is the normal force.
2. Normal modes:

- When all mobile parts of a system vibrate at the same frequency.
- Normal mode ansatz: \( q_i = q_{i0} e^{i\omega t} \), plug into the EoM to get a matrix equation \( \Omega q = 0 \). For nontrivial solutions, we require \( \det(\Omega) = 0 \), which gives a polynomial equation for \( \omega \) whose real solutions are the normal mode frequencies.
- Relative amplitude and phase: plug in the solutions for \( \omega \) into \( \Omega \). The amplitudes \( q_{i0} \) can all be written as some (possibly complex) number multiplying one of the amplitudes (they’re all proportional to each other). This tells you their relative sizes and phases.
- There ought to be as many normal modes as generalized coordinates, but some of them may very well have the same frequency.
- Any motion of the system can be expanded in terms of normal modes.

3. Fictitious forces: \( \mathbf{F}_{\text{Cor}} = -2m\omega \times \mathbf{v} \) and \( \mathbf{F}_{\text{cent}} = -m\omega \times (\omega \times \mathbf{r}) \).

4. Bernoulli’s equation: \( P + \frac{1}{2} \rho v^2 + \rho gh = \text{constant} \).

5. Special relativity (set \( c = 1 \)):

- Invariant interval: \( \Delta s^2 = -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 \).
- Relation between proper time and coordinate time: \( dt = \gamma d\tau \).
- 4-position \( x^\mu = (t, x, y, z) \) and 4-momentum \( p^\mu = m\frac{dx^\mu}{d\tau} = \gamma m\dot{x}^\mu \). Notice that the \( p^0 = \gamma m = E \), the energy.\( E^2 = m^2 + p^2 \).
- Boost along \( x \)-direction (left) and \( y \)-direction (right):

\[
\Lambda^\mu_\nu = \begin{pmatrix}
\gamma & -\beta \gamma & 0 & 0 \\
-\beta \gamma & \gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad \Lambda^\mu_\nu = \begin{pmatrix}
\gamma & 0 & -\beta \gamma & 0 \\
0 & 1 & 0 & 0 \\
-\beta \gamma & 0 & \gamma & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

Hopefully, it’s clear how to write a boost along the \( z \)-axis. I would never boost along a mixed axis; I would rotate the axes first so that the boost axis coincides with one of the Cartesian axes. Then boost.

- Figure 2.1 shows Minkowski diagram of a frame, \( S' \), moving relative to another, \( S \). In the diagram, \( \beta = 3/5 \) and \( \gamma = 5/4 \). So, when 4 seconds pass in \( S' \) from the point of view of \( S \), 5 seconds pass in \( S \) from \( S' \)’s own point of view. That’s time dilation. Can you explain length contraction via the diagram? The \( t' \) and \( x' \) axes are reflections of each other about the positive light cone (black dashed line with slope = 1). The slopes of the \( t' \) and \( x' \) axes are \( 1/\beta \) and \( \beta \), respectively.

\[
\text{Figure 2.1: Minkowski diagram of a moving frame.}
\]
6. Central forces and Kepler:

-Angular momentum is conserved.

\[ E = \frac{1}{2} m r^2 + V_{\text{eff}}(r) \text{ where } V_{\text{eff}}(r) = \frac{L^2}{2mr^2} + V(r). \]

-For gravity \((V(r) = -\alpha/r)\): \( k = r(1 + \epsilon \cos \theta) \text{ where } k = L^2/m\alpha \text{ and } \epsilon = \left(1 + \frac{2EL^2}{m\alpha^2}\right)^{1/2} \) is the eccentricity.

-Circular orbit: \( \epsilon = 0 \) and \( E = -m\alpha^2/2L^2 \).

-Elliptical orbit: \( 0 < \epsilon < 1 \) and \(-m\alpha^2/2L^2 < E < 0\).

-Parabola: \( \epsilon = 1 \) and \( E = 0 \).

-Hyperbola: \( \epsilon > 1 \) and \( E > 0 \).

\[ \begin{align*}
V_{\text{eff}} & \quad \text{Hyperbola} \\
\quad \text{Parabola} \\
\quad \text{Ellipse} \\
\quad \text{Circle}
\end{align*} \]

Figure 2.2: Central forces energy and effective potential energy diagram.

-Kepler’s first law: planets move in elliptical orbits with the sun at one focus.

-Kepler’s second law: the radius vector to a planet sweeps out area at a rate that is independent of its position in the orbit.

-Kepler’s third law: \( T = 2\pi/\dot{\phi}, \) which in the case of Newtonian gravity gives \( T^2 = 4\pi^2a^3/GM_\odot, \) where \( T \) is the period of orbit, \( a \) is the semi-major axis and \( M_\odot \) is the mass of the sun (or whatever else produces the central potential).
2.1 Unrolling Spool

[Fall 2005 Classical (Morning), Problem 1] A large spool of rope of mass $M$ stands on the ground with the end of the rope lying on the top edge of the spool. The radius of the spool is $R$. A person grabs the end of the rope and walks a distance $L$. The spool does not slide.

(a) What length of the rope unwinds from the spool?

(b) How far does the spool’s center of mass move?

(c) At one point the spool will hit a step of height $h$, against which the spool rests. What force should the person apply in order for the spool to climb the step? (You can ignore inertia in this final part, i.e. the spool doesn’t jump the curb simply because it’s moving fast.)

Figure 2.3: Fall 2005 Classical (Morning), Problem 1

SOLUTION:

(a) Let a length $x$ of rope unwind. Then, the spool has rotated through $\theta = x/R$. Since the spool does not slip, its center of mass has moved a distance $R\theta = x$. The sum of the distance the spool moves and the amount of rope that unwinds is the distance, $L$, travelled by the person. Hence, $2x = L$ and so $x = L/2$.

(b) $x = L/2$ by part (a).

(c) Below is a free body diagram for the spool. There are two normal forces, one from the ground and one from the step, gravity and the pulling force.

\[ F_{p\perp} = \frac{R - h}{R} F_p, \quad F_{g\perp} = \frac{\sqrt{2Rh - h^2}}{R} Mg. \] (2.1.1)
Right when the spool rises, $F_n$ (the one from the ground) vanishes. Thus, letting the perpendicular forces just about cancel at this point means $F_{p\perp} = F_{g\perp}$ and solving for $F_p$ yields

$$F_p = \left( \frac{\sqrt{2Rh - h^2}}{R - h} \right) Mg \quad (2.1.2)$$

You may want to check out some of the limiting cases. For example, it should be physically clear that there is no solution when $h > R$ as the above answer indeed corroborates (it’s negative in this range.)
2.2 Another Spool

[Morin 2.17] A spool of mass $M$ consists of an axle of radius $r$ and an outside circle of radius $R$ which rolls without slipping on the ground. A thread is wrapped around the axle and is pulled with tension $T$ at an angle $\theta$ as shown.

(a) Given $R$ and $r$, what should $\theta$ be so that the spool doesn’t move?

(b) Given $R$, $r$ and the coefficient of friction $\mu$ between the spool and the ground, what is the largest value of $T$ for which the spool remains at rest?

(c) Given $R$ and $\mu$, what should $r$ be so that you can make the spool slip from the static position with as small a $T$ as possible? What is the resulting value of $T$?

\[
\begin{align*}
\text{Figure 2.5: Another spool.}
\end{align*}
\]

SOLUTION:

(a) The diagram below shows the configuration at which all the forces (tension, gravity, normal and friction) are directed through the point of contact between the spool and the ground. This guarantees zero torque and thus no rotation.

\[
\begin{align*}
\text{Figure 2.6: Configuration with vanishing torque.}
\end{align*}
\]

Simple trigonometry gives

\[
\theta = \cos^{-1}(r/R),
\]

(2.2.1)

Note: you can also get this by balancing the horizontal forces, $T \cos \theta = F_f$, and torques around the center of the spool, $Tr = F_f R$.

(b) Since the spool remains at rest, the value of $\theta$ is given by part (a). Balancing vertical forces gives $N = Mg - T \sin \theta$. This gives

\[
T \cos \theta = F_f \leq \mu N = \mu(Mg - T \sin \theta).
\]

(2.2.2)

Thus, we have

\[
T \leq \frac{\mu Mg}{\cos \theta + \mu \sin \theta} = \frac{\mu Mg R}{r + \mu \sqrt{R^2 - r^2}}
\]

(2.2.3)
(c) We would like to minimize the value of $T$ from part (b) with respect to $r$. We can maximize the denominator with respect to $\theta$ to get $\tan \theta = \mu$, which we can plug back in to get the resulting value of $T$:

$$T_0 = \frac{\mu Mg}{\sqrt{1 + \mu^2}} \quad (2.2.4)$$

Right at the threshold of slipping, $\theta$ and $r$ are still related via (2.2.1). Thus, $\tan \theta = \sqrt{R^2 - r^2}/r = \mu$ gives

$$r_0 = \frac{R}{\sqrt{1 + \mu^2}} \quad (2.2.5)$$
2.3 Rope Wrapped Around a Pole

[Kevin G.] You hold one end of a rope which wraps around a pole and whose other end is attached to a boat at the docks. You pull with a tension $T_0$ and the current causes the boat to tug at the rope with tension $e^\alpha T_0$, that is $e^\alpha$ times stronger than you can pull, for some number $\alpha$. Let $\mu$ be the coefficient of friction between the rope and the pole. What is the minimum number of times the rope must wind around the pole in order for it not to slip (does not need to be a whole number)?

**SOLUTION:**

Below is a free body diagram for a small piece of rope that subtends a differential angle $d\theta$. There are tensions on either end and a normal force which balances the horizontal components of the tensions. The marked angles are all equal.

![Free Body Diagram](image)

Therefore, balancing horizontal forces yields

$$dN = (T + dT) \sin(d\theta/2) + T \sin(d\theta/2) \approx T \, d\theta + \frac{1}{2} dT \, d\theta.$$  \hspace{1cm} (2.3.1)

Balancing vertical forces yields

$$T + dT \approx (T + dT) \cos(d\theta/2) = T \cos(d\theta/2) + df \approx T + df.$$  \hspace{1cm} (2.3.2)

Finally, combining these gives

$$dT = df \leq \mu dN = \mu T \, d\theta \quad \Rightarrow \quad \frac{dT}{d\theta} \leq \mu T.$$  \hspace{1cm} (2.3.3)

Equality is solved by an exponential starting with $T_0$ at your end ($\theta = 0$):

$$T \leq e^{\mu \theta} T_0.$$  \hspace{1cm} (2.3.4)

We want $T$ at the other end to be equal to $e^\alpha T_0$. So, we want to solve for $\theta$ in

$$e^{\alpha T_0} \leq e^{\mu \theta} T_0 \quad \Rightarrow \quad \theta \geq \alpha / \mu.$$  \hspace{1cm} (2.3.5)

The number of windings is $n_{\text{wind}} = \theta / 2\pi$ and so

$$n_{\text{wind}} \geq \frac{\alpha}{2\pi \mu}.$$  \hspace{1cm} (2.3.6)
2.4 Two Masses and Three Springs

[Fall 2008 Classical (Afternoon), Problem 5] Two masses, \( m_1 = m, m_2 = 2m \), are arranged into a 1D coupled harmonic oscillator using three identical springs of spring constant \( k \), as shown in the diagram below. At equilibrium positions, all of the springs are at their normal lengths (i.e. not stretched). Assume these masses are confined to oscillate in the \( x \)-direction.

(a) Write down the Lagrangian for this system.

(b) Derive the equations of motion for the two masses.

(c) Determine the normal mode frequencies.

(d) For each normal mode, derive the relative amplitude and phase of the oscillations of the two masses.

(e) Describe and/or sketch the motion of each normal mode.

(f) At \( t = 0 \), the masses are started from rest, but with \( m_1 \) displaced a distance \( x_1 = D \) to the right. Derive the motion of the two masses as a function of \( t \).

Figure 2.8: Fall 2008 Classical (Afternoon), Problem 5

SOLUTION:

(a) The kinetic energies are obvious. The potential energy in the left spring is \( \frac{1}{2}kx_1^2 \), that in the right spring is \( \frac{1}{2}kx_2^2 \), and that in the middle spring is \( \frac{1}{2}k(x_2 - x_1)^2 \). Adding these gives \( k(x_1^2 - x_1x_2 + x_2^2) \). Hence, the Lagrangian is

\[
L = \frac{1}{2}m(\dot{x}_1^2 + 2\dot{x}_2^2) - k(x_1^2 - x_1x_2 + x_2^2)
\]

(b) The E-L EoM for \( x_1 \) and \( x_2 \) are

\[
\ddot{x}_1 + 2\omega_0^2(x_1 - \frac{1}{2}x_2) = 0 \quad \ddot{x}_2 + \omega_0^2(x_2 - \frac{1}{2}x_1) = 0
\]

where \( \omega_0^2 = k/m \).

(c) Posit a normal mode ansatz: \( x_1 = Ae^{i\omega t} \) and \( x_2 = Be^{i\omega t} \). Then, the equations of motion can be written in matrix form:

\[
\begin{pmatrix}
2\omega_0^2 - \omega^2 & -\omega_0^2 \\
-\omega_0^2/2 & \omega_0^2 - \omega^2
\end{pmatrix}
\begin{pmatrix}
A \\
B
\end{pmatrix}
=
\begin{pmatrix}
0 \\
0
\end{pmatrix}
\]

In order for there to be nontrivial solutions for \( A \) and \( B \), the determinant of the left matrix must vanish; otherwise, we could simply invert the equation, implying \( A = B = 0 \). This gives the equation

\[
\omega^4 - 3\omega_0^2\omega^2 + \frac{3}{2}\omega_0^4 = 0
\]

Solving for \( \omega^2 \) yields two normal modes, \( \omega_\pm \), given by

\[
\omega^2_\pm = \frac{3}{2}\omega_0^2 \pm \sqrt{\frac{9}{4}\omega_0^4 - \frac{3}{2}\omega_0^4} = \frac{3}{2}\omega_0^2 \pm \frac{3\sqrt{3}}{2}\omega_0^2
\]
(d) Plug these frequencies back into (2.4.3) and factoring out $\omega_0^2/2$ gives
\[
\begin{pmatrix}
1 \mp \sqrt{3} & -2 \\
-1 & -1 \mp \sqrt{3}
\end{pmatrix}
\begin{pmatrix}
A_+ \\
B_+
\end{pmatrix} = \begin{pmatrix}
0 \\
0
\end{pmatrix}.
\] (2.4.6)

These necessarily give the same solutions, but we’ll using the first one to solve for $B_\pm$ in terms of $A_\pm$:
\[
B_\pm = \frac{1\mp \sqrt{3}}{2} A_\pm.
\] (2.4.7)

(e) The “+” mode has $B_+ = \frac{1-\sqrt{3}}{2} A_+$; the proportionality is negative and thus the two masses oscillate 180° out of phase. Since $|B_+| < |A_+|$, the larger mass, $m_2$, oscillates at a lower amplitude than the smaller mass, $m_1$.

The “−” mode has $B_− = \frac{1+\sqrt{3}}{2} A_−$; the proportionality is positive and thus the two masses oscillate in phase. In this mode, the larger mass has a higher amplitude than the smaller mass.

(f) Since we have all of the normal modes, any motion of the system can be expanded in terms of our normal modes. We write
\[
\begin{pmatrix}
x_1(t) \\
x_2(t)
\end{pmatrix} = A_+ \left( 1 - \frac{1}{\sqrt{3}} \right) e^{i\omega_+ t} + A_- \left( 1 + \frac{1}{\sqrt{3}} \right) e^{i\omega_- t}.
\] (2.4.8)

At $t = 0$, we can massage these two equations into
\[
A_+ + A_- = D, \quad A_+ - A_- = \frac{\sqrt{3}}{3} D.
\] (2.4.9)

Solving for the two constants yields
\[
A_+ = \frac{3+\sqrt{3}}{6} D, \quad A_- = \frac{3-\sqrt{3}}{6} D.
\] (2.4.10)

Thus, the motion is given by
\[
\begin{align*}
x_1(t) &= \frac{3+\sqrt{3}}{6} De^{i\omega_+ t} + \frac{3-\sqrt{3}}{6} De^{i\omega_- t} \\
x_2(t) &= -\frac{\sqrt{3}}{6} De^{i\omega_+ t} + \frac{\sqrt{3}}{6} De^{i\omega_- t}
\end{align*}
\] (2.4.11)
2.5 Circular Spring and Masses

[Kevin G.] Three identical masses are interconnected via three identical springs and the system is constrained to move along a hoop. Find the normal modes. Suppose you displace the top mass a bit and the masses are initially at rest, determine the subsequent motion of the system after letting go of the top mass.

Figure 2.9: Circular Spring and Masses.

SOLUTION:

Since there are three free masses and each one is constrained to move along one dimension, there should be exactly three normal modes. This is a sufficiently small number that we can probably just intuit them. One normal mode is when the whole system of springs and masses just rotate together. The other two are when one mass sits still and the other two vibrate oppositely:

Figure 2.10: Circular spring normal mode.

Let the masses be labeled 1, 2 and 3 starting with 1 as the top mass in Figure 2.9 and going round clockwise. Let \( x_i \) be the displacement away from equilibrium of the \( i \)th mass around the circle (clockwise is positive.) Then, the first normal mode is

\[
x_i = At + B,
\]

for some constants \( A \) and \( B \). We could set \( B = 0 \) by demanding that \( x_i(0) = 0 \).

For the normal mode depicted in Figure 2.10, we set \( x_2 = -x_3 = x \). For each mass, the stretched spring pulls with force \( kx \) and the compressed spring pushes with force \( 2kx \) (since it is compressed twice as much as each one of the other two springs is stretched.) This gives a total restoring force of \( 3kx \) and so the frequency of oscillation is \( 3\omega_0^2 \), where \( \omega_0^2 = k/m \). Let the other normal mode be when mass 2 is stationary. We can write, where \( i \) is the index of the other mass (not \( m_i \)) that is moving,

\[
x_i = \epsilon_{i2} C_1 \cos(\sqrt{3} \omega_0 t + \phi_1), \quad \text{and} \quad x_i = \epsilon_{i3} C_2 \cos(\sqrt{3} \omega_0 t + \phi_2),
\]

(2.5.2)

One will have to be quite creative using this method when there are more masses. So, let us get some practice using the determinant method. The \( F = ma \) equations for the three masses are

\[
m\ddot{x}_1 - k(x_2 - x_1) - k(x_3 - x_1) = 0,
m\ddot{x}_2 - k(x_3 - x_2) - k(x_1 - x_2) = 0, \quad \text{and} \quad m\ddot{x}_3 - k(x_1 - x_3) - k(x_2 - x_3) = 0.
\]

(2.5.3)
Posit a normal mode ansatz (all masses move with the same frequency): \( x_i = A_i e^{i\omega t} \). Then, (2.5.3) can be put in matrix form:

\[
\begin{pmatrix}
\omega^2 - 2\omega_0^2 & \omega_0^2 & \omega_0^2 \\
\omega_0^2 & \omega^2 - 2\omega_0^2 & \omega_0^2 \\
\omega_0^2 & \omega_0^2 & \omega^2 - 2\omega_0^2
\end{pmatrix}
\begin{pmatrix}
A_1 \\
A_2 \\
A_3
\end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
\]

(2.5.4)

You can use Sarrus’ rule or expansion by minors or whatever other method to compute the determinant of the big matrix on the left. In order for there to be nontrivial solutions for \( A_i \), the determinant must vanish, otherwise we could just invert the equation and get \( A_i = 0 \). After a bit of algebra, one finds the equation

\[
\omega^2(\omega^2 - 3\omega_0^2)^2 = 0,
\]

(2.5.5)
to which the solutions are the ones we found earlier: \( \omega = 0 \) and \( \omega = \sqrt{3} \omega_0 \). The solution \( \omega = 0 \) does not correspond to \( x_i = A_i \), a constant. It just means that the masses don’t oscillate - they just rotate together. In this case, their displacements are given by (2.5.1).

The next step is to find the amplitudes for the other two modes. Plugging in the value \( \omega^2 = 3\omega_0^2 \) turns the big matrix into \( \omega_0^2 \) multiplying the matrix with 1 in each entry. Then, the three equations are all the same and just imply \( A_1 + A_2 + A_3 = 0 \). Since there are three constants and one relation, there are two independent solutions. We choose to set one to 0 and the other two to \( \pm C \). Of course, we could have put an arbitrary phase in the initial ansatz: \( x_i = A_i e^{i(\omega t + \phi)} \). Our choice gives us the two normal modes

\[
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
= C_1 \begin{pmatrix} 0 & 1 & -1 \end{pmatrix} e^{i \sqrt{3} \omega_0 t + \phi_1},
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
= C_2 \begin{pmatrix} 1 & 0 & -1 \end{pmatrix} e^{i \sqrt{3} \omega_0 t + \phi_2}.
\]

(2.5.6)
The real parts, of course, agree with (2.5.2).

Write the motion as a superposition of normal modes:

\[
\begin{pmatrix}
x_1(t) \\
x_2(t) \\
x_3(t)
\end{pmatrix}
= (At + B) \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}
+ C_1 e^{i\phi_1} e^{i\sqrt{3} \omega_0 t} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}
+ C_2 e^{i\phi_2} e^{i\sqrt{3} \omega_0 t} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.
\]

(2.5.7)

At \( t = 0 \), this reads, where \( \epsilon \) is the displacement of \( m_1 \),

\[
\begin{pmatrix}
\epsilon \\
0 \\
0
\end{pmatrix}
= B \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}
+ C_1 e^{i\phi_1} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}
+ C_2 e^{i\phi_2} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},
\]

(2.5.8)

whose solution is \( \epsilon / 3 = B = -C_1 e^{i\phi_1} = C_2 e^{i\phi_2} / 2 \).

Thus, the general solution now reads

\[
\begin{pmatrix}
x_1(t) \\
x_2(t) \\
x_3(t)
\end{pmatrix}
= At \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}
+ \frac{\epsilon}{3} \begin{pmatrix} 1 & 2 e^{i\sqrt{3} \omega_0 t} & 1 - e^{i\sqrt{3} \omega_0 t} \\ 1 + 2 e^{i\sqrt{3} \omega_0 t} & 1 - e^{i\sqrt{3} \omega_0 t} \\ 1 - e^{i\sqrt{3} \omega_0 t} & 1 + e^{i\sqrt{3} \omega_0 t} \end{pmatrix}.
\]

(2.5.9)

Taking the real part yields

\[
\begin{pmatrix}
x_1(t) \\
x_2(t) \\
x_3(t)
\end{pmatrix}
= At \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}
+ \frac{\epsilon}{3} \begin{pmatrix} 1 + 2 \cos(\sqrt{3} \omega_0 t) & 1 - \cos(\sqrt{3} \omega_0 t) \\ 1 - \cos(\sqrt{3} \omega_0 t) & 1 + 2 \cos(\sqrt{3} \omega_0 t) \\ 1 + \cos(\sqrt{3} \omega_0 t) & 1 - \cos(\sqrt{3} \omega_0 t) \end{pmatrix}.
\]

(2.5.10)
Now, setting the time derivatives at $t = 0$ to zero implies $A = 0$. Hence,

\[
\begin{pmatrix}
    x_1(t) \\
    x_2(t) \\
    x_3(t)
\end{pmatrix} = \frac{\epsilon}{3} \begin{pmatrix}
    1 + 2\cos(\sqrt{3}\omega_0 t) \\
    1 - \cos(\sqrt{3}\omega_0 t) \\
    1 - \cos(\sqrt{3}\omega_0 t)
\end{pmatrix}.
\]  

(2.5.11)
2.6 Seven Pool Balls

[Morin 5.76] Seven identical pool balls are situated at rest as shown below. The middle ball suddenly somehow acquires a speed $v$ to the right. Assume that starting with ball $A$, the balls spiral out an infinitesimal amount. So $A$ is closer to the center ball than $B$ is, and $B$ is closer that $C$ is, etc. This means that the center ball collides with $A$ first, then it gets deflected into $B$, and then it gets deflected into $C$, and so on. But all these collisions happen in the blink of an eye. What will the center ball’s velocity be after it collides (elastically) with all six balls?

![Figure 2.11: Seven pool balls.](image)

**SOLUTION:**

When identical masses collide elastically (and not head on), the angle between their deflected velocities is always 90°. This is a consequence of conservation of momentum and energy (try to prove it.) We will measure angles from directly east (horizontal to the right) with positive going counterclockwise. Let $v_i$ be the speed of the center ball after the $i^{th}$ collision, and $v_0 = v$.

Geometry dictates that $A$ is deflected at 30° and so the center ball is deflected at −60° and conservation of momentum dictates that $v_1 = v_0 \sin 30° = v/2$. Now, −60° is directly between $B$ and $C$, so the process repeats itself: after colliding with $B$, the center ball is deflected to the space between $C$ and $D$ with speed $v_2 = v_1 \sin 30° = v/4$, and so on. Thus, the final velocity of the center ball is $v/64$ to the right.
2.7 Colliding Photons

[Goldstein, Poole & Safko 7.22] A photon of energy $\mathcal{E}$ collides at angle $\theta$ with another photon of energy $E$. Determine the minimum value of $\mathcal{E}$ permitting the formation of a pair of particles of mass $m$.

**Figure 2.12: Colliding photons.**

**SOLUTION:**

Set $c = 1$ for the moment. Let the four-momenta of the photons be

$$p_1 = E(1, 1, 0, 0), \quad p_2 = \mathcal{E}(1, \cos \theta, \sin \theta, 0). \quad (2.7.1)$$

Rotate in the $xy$-plane so that the total momentum is in the $+x$ direction:

$$p_t = (E + \mathcal{E}, \sqrt{E^2 + \mathcal{E}^2 + 2E\mathcal{E}\cos \theta}, 0, 0)$$

$$= (E + \mathcal{E}) \left(1, \sqrt{1 - \frac{2E\mathcal{E}(1 - \cos \theta)}{(E + \mathcal{E})^2}}, 0, 0\right). \quad (2.7.2)$$

Now, boost in the $+x$ direction by some amount $\beta$:

$$\gamma \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \left( E + \mathcal{E} \right) \left( \frac{1}{\sqrt{\cdots}} \right) = \gamma (E + \mathcal{E}) \begin{pmatrix} 1 - \beta \sqrt{\cdots} \\ -\beta + \sqrt{\cdots} \end{pmatrix}, \quad (2.7.3)$$

where $\cdots$ indicates the terms in the square root in (2.7.2).

To make the boosted frame the center of momentum frame, we need the $x$ component of the boosted total momentum to vanish, which implies $\beta = \sqrt{\cdots}$. The corresponding $\gamma$ factor is

$$\gamma = (1 - \beta^2)^{-1/2} = \left(1 - 1 + \frac{2E\mathcal{E}(1 - \cos \theta)}{(E + \mathcal{E})^2}\right)^{-1/2} = \frac{E + \mathcal{E}}{\sqrt{2E\mathcal{E}(1 - \cos \theta)}}. \quad (2.7.4)$$

Thus, the total energy in the center of momentum frame is

$$E_{\text{com}} = \gamma (E + \mathcal{E})(1 - \beta^2) = \sqrt{2E\mathcal{E}(1 - \cos \theta)}. \quad (2.7.5)$$

This must be $\geq 2m$, which, after reintroducing $c$, yields the inequality

$$\mathcal{E} \geq \frac{2mc^4}{E(1 - \cos \theta)}. \quad (2.7.6)$$
2.8 Pion Photoproduction

[Lim 3027] Consider the pion photoproduction reaction $\gamma + p \rightarrow p + \pi^0$, where the rest energy is 938 MeV for the proton and 135 MeV for the neutral pion.

(a) If the initial proton is at rest in the laboratory, find the laboratory threshold gamma-ray energy for this reaction to go.

(b) The isotropic 3 K cosmic blackbody radiation has an average photon energy of about $10^{-3}$ eV. Consider a head-on collision between a proton and a photon of energy $10^{-3}$ eV. Find the minimum proton energy that will allow this pion photoproduction reaction to go.

(c) Speculate briefly on the implications of your result for part (b) for the energy spectrum of cosmic-ray protons.

SOLUTION:

(a) Consider the initial value of $E^2 - (pc)^2$, which, calculated in the lab frame is $(E_{\gamma} + m_p c^2)^2 - (\frac{E_{\gamma}}{c})^2 = 2E_{\gamma}m_p c^2 + (m_p c^2)^2$. Since this is an invariant, it is the same in any reference frame. At threshold, in the center of mass frame, the resulting proton and pion are at rest. Thus, the final value of $E^2 - (pc)^2$ is $(m_p + m_\pi)^2 c^4 = (m_p c^2)^2 + 2m_p m_\pi c^4 + (m_\pi c^2)^2$. Again, this is frame-independent. Solving for $E_{\gamma}$ gives

$$E_{\gamma} = m_\pi c^2 \left( 1 + \frac{m_\pi}{2m_p} \right) = 145 \text{ MeV}.$$  \hspace{1cm} (2.8.1)

(b) As explained by Victoria/Weinberg, head-on means that the photon and proton have oppositely directed momenta, but implies nothing about the COM frame. I will take the energy of the photon in the lab frame to be $E_{\gamma} = 10^{-3}$ eV. Again, equating the initial and final values of $E^2 - (pc)^2$ yields

$$(\gamma m_p c^2 + E_{\gamma})^2 - \left( \frac{E_{\gamma}}{c} - \gamma m_p \beta c \right)^2 c^2 = (m_p + m_\pi)^2 c^4.$$  \hspace{1cm} (2.8.2)

Solving for $\beta$ gives

$$\gamma(1 + \beta) = \sqrt{1 + \frac{1 + \beta}{1 - \beta}} = \frac{m_\pi c^2}{E_{\gamma}} \left( 1 + \frac{m_\pi}{2m_p} \right) = 1.447 \times 10^{11}.$$  \hspace{1cm} (2.8.3)

Since $\gamma(1 + \beta)$ is so large, $\beta$ must be very close to 1. Let us solve approximately for $\gamma$ by setting $\beta = 1$ in (2.8.3). This gives $\gamma = 7.235 \times 10^{10}$ and so the proton energy is

$$E_p = \gamma m_p c^2 = 7.235 \times 10^{10} \times 0.938 \text{ MeV} = 6.787 \times 10^{10} \text{ GeV}.$$  \hspace{1cm} (2.8.4)

(c) This implies that the energy spectrum of cosmic ray protons with $E > 6.79 \times 10^{10}$ GeV will be depleted to some degree due to interaction with the cosmic blackbody radiation.
2.9 W Bosons from Electron-Positron Collisions

[Fall 1994 Classical (Morning), Problem 6] At a future linear position-electron ($e^+e^-$) collider, electron and positron beams have equal energy $E$ but opposite momenta. (CDF and DØ announced the discovery of the top the next year, in 1995.) Production of top quark-antiquark pairs ($tt\bar{t}$) is expected to occur via the reaction

$$e^+e^- \rightarrow t + \bar{t},$$

where $E$ is not necessarily equal to $m_t$. Very soon thereafter, the top quarks decay according to

$$t \rightarrow W^+ + b,$$

$$\bar{t} \rightarrow W^- + \bar{b}.$$ 

Here, you may neglect all rest masses except for those of the $t$ and $W$ (particle and antiparticle masses are the same); for the purposes of this problem, you may approximate $m_t = 2m_W$.

Under all circumstances, it would be highly improbable for one of the $W$’s to be produced completely at rest in the laboratory. But is it kinematically possible? If so, what restriction, if any, is placed on $E$ (expressed in terms of $m_t$)?

**SOLUTION:**

**Method 1:** Set $c = 1$. Let $t^\mu, W^\mu$ and $b^\mu$ be the 4-momenta of the corresponding particles. Energy-momentum conservation implies $t^\mu = W^\mu + b^\mu$. Consider the $t$ rest frame, wherein we denote the 4-momenta with stars. In this frame, $t^* = (m_t, 0)$. For any 4-momentum, $q^\mu$, we have

$$q \cdot q = E^2 + p^2 = m^2.$$ 

Thus,

$$b^* \cdot b^* = t^* \cdot t^* - 2t^* \cdot W^* + W^* \cdot W^*$$

$$m_b^2 = m_t^2 - 2m_t^* E_W^* + 2(0 \cdot p_W^*) + m_W^2$$

$$0 \approx 5m_W^2 - 4m_W E_W^*.$$ (2.9.1)

Hence, the the $t$ rest frame,

$$\gamma_W^* = \frac{E_W^*}{m_W^*} = \frac{5}{4},$$ (2.9.2)

The lab frame is the COM frame for the $e^+e^-$ and $tt\bar{t}$ pairs. Thus, in this frame, $E_t = E$ and so

$$\gamma_t = \frac{E_t}{m_t} = \frac{E}{m_t},$$ (2.9.3)

The $W$ can only be at rest in lab frame if its speed in the $t$ rest frame is equal and opposite to the speed of the $t$ in the lab frame. Thus, $\gamma_t = \gamma_W^*$, which implies

$$E = \frac{5}{4}m_t c^2.$$ (2.9.4)

**Method 2:** Momentum conservation implies $p_b^* = -p_W^*$, and thus $p_b^* = p_W^*$. Energy conservation implies $m_t = E_W^* + E_b^*$. Masslessness of $b$ implies $m_t \approx E_W^* + p_b^* = E_W^* + p_W^*$. Thus,

$$\sqrt{(p_W^*)^2 + m_W^2} = E_W^* = m_t - p_W^* = 2m_W - p_W$$

$$(p_W^*)^2 + m_W^2 = 4m_W^2 - 4m_W p_W^* + (p_W^*)^2$$

$$p_W^* = \frac{3}{4}m_W.$$ (2.9.5)

This implies

$$E_W^* = 2m_W - p_W^* = \frac{5}{2}m_W.$$ (2.9.6)

Finally, $\gamma_W^* = E_W^*/m_W = 5/4$ and everything follows as in Method 1.
2.10 Virtual Exchanged Particles

[Kevin G.] In Quantum Electrodynamics, the Coulomb force between charged point particles arises via the exchange of a photon as in the diagram below, where time flows upward. In Quantum Chromodynamics, the strong force between quarks arises via the exchange of a gluon. Real photons and gluons are massless. It is often said that these exchanged photons and gluons have mass and are called virtual. Show that, in fact, if the exchanged particles are to have a mass, it would necessarily be imaginary, which, more than anything, is the “reason” they’re called virtual.

![Virtual exchanged particles](image)

**Figure 2.13: Virtual exchanged particles.**

**SOLUTION:**

Let $p_i$ and $p'_1$ be the incoming and outgoing 4-momenta of the two particles and let $q$ be the 4-momentum of the exchanged particle. Conservation of 4-momentum implies $p'_1 = p_1 + q$ and $p'_2 = p_2 - q$ (the sign of $q$ is not crucial; its appearances in $p'_1$ and $p'_2$ just need to have opposite signs.)

We have the 3-momentum relation, $q = p'_1 - p_1$, which implies $|q|^2 = |p'_1|^2 + |p_1|^2 - 2p_1 \cdot p'_1$.

Let $m_1$ be the mass of particle 1 and $m$ be the mass of the exchanged particle. Then, letting $\theta$ be the angle between $p_1$ and $p'_1$, we have

$$m^2 \gamma^2 \beta^2 = m_1^2 \left[ \gamma_1^2 \beta_1^2 + \gamma_1^2 \beta_1^2 - 2\gamma_1 \gamma'_1 \beta_1 \beta'_1 \cos \theta \right]. \quad (2.10.1)$$

We also have the energy relation $E = E'_1 - E_1$, where $E$ is the energy of the exchanged particle. Thus, $m\gamma = m_1(\gamma'_1 - \gamma_1)$, or

$$m^2 \gamma^2 = m_1^2 \left[ \gamma_1^2 + \gamma_1^2 - 2\gamma_1 \gamma'_1 \right]. \quad (2.10.2)$$

Subtract (2.10.1) from (2.10.2) and use $\gamma^2(1 - \beta^2) = 1$ to get

$$m^2 = 2m_1^2 \left[ 1 - \gamma_1 \gamma'_1 + [(\gamma_1^2 - 1)(\gamma'_1^2 - 1)]^{1/2} \cos \theta \right]. \quad (2.10.3)$$

Denote by $A$ the factor multiplying $2m_1^2$. Then,

$$A \leq B \equiv 1 - \gamma_1 \gamma'_1 + [(\gamma_1^2 - 1)(\gamma'_1^2 - 1)]^{1/2}. \quad (2.10.4)$$

Now, certainly, $-(\gamma_1 - \gamma'_1)^2 < 0$ and so $-\gamma_1^2 - \gamma'_1^2 < -2\gamma_1 \gamma'_1$. Add $\gamma_1^2 \gamma'_1 + 1$ to both sides of the inequality to get $(\gamma_1^2 - 1)(\gamma'_1^2 - 1) < (\gamma_1^2 - 1)^2$, which, when square rooted implies $B < 0$. It follows that $A < 0$ and so $m^2 < 0$, i.e. the exchanged particle is virtual since it has imaginary mass.
2.11 Moving Sound Reflector

[Kevin G.] Sound waves of speed $v$ and frequency $\omega$ are incident on a perfectly reflecting wall that is moving at speed $\beta c$ towards the source. Show that the observer in the lab frame will hear beats. What is the observed beat frequency? If $\omega$ is around the bottom of the human audible range, then how fast must the wall move in order that the observed sound be around the top of the audible range?

**SOLUTION:**

Let the wave come in from the positive $x$-axis towards the origin. It can be described by

$$\psi_i(x, t) = \sin[\omega(\frac{x}{v} + t)].$$

Next, we must compute the outgoing frequency. The incoming 4-vector is $k^\mu_i = \frac{\omega}{v}(1, -1, 0, 0)$. We must boost in the positive $x$-direction (negatives on the off-diagonals): $k^\mu_i = \gamma(1 + \beta)\frac{\omega}{v}(1, -1, 0, 0)$. In this frame, the reflected wave is simply $k^\mu_r = \gamma(1 + \beta)\frac{\omega}{v}(1, 1, 0, 0)$. We boost this back in the negative $x$-direction (positives on the off-diagonals): $k^\mu_r = \gamma^2(1 + \beta)^2\frac{\omega}{v}(1, 1, 0, 0) = \frac{1 + \beta}{1 - \beta}\frac{\omega}{v}(1, 1, 0, 0)$. Therefore, the frequency of the reflected wave in the lab frame is $\omega_r = \alpha\omega$, where $\alpha = \frac{1 + \beta}{1 - \beta}$. The reflected wave is $\psi_r(x, t) = \sin[\alpha\omega(\frac{x}{v} - t)]$.

To get the total wave, we just add the incident and reflected waves. Using $\sin u + \sin v = 2\sin[\frac{u}{2}]\cos[\frac{v}{2}]$, we get

$$\psi_{tot}(x, t) = 2\sin[\frac{\omega + \omega_r}{2}\frac{x}{v} + \frac{\omega - \omega_r}{2}t]\cos[\frac{\omega + \omega_r}{2}\frac{x}{v} + \frac{\omega - \omega_r}{2}t]$$

$$= 2\sin[\frac{\beta\omega}{1 - \beta}(\frac{x}{v} - t)]\cos[\frac{\omega}{1 - \beta}(\frac{x}{v} - t)].$$

The frequency of the sine part is $\frac{\omega}{1 - \beta}$ and that of the cosine part is $\frac{\omega}{1 + \beta}$. Since $\beta \leq 1$, the cosine part always has the higher frequency. The speed of the sine part is $\beta v$ and that of the cosine part is $v/\beta$. Again, the cosine part is faster. These two things imply that the cosine part has the longer wavelength, and is thus the envelope enclosing the much shorter wavelength component, which is the sine. Thus, the cosine part describes the beats and thus the beat frequency is $\omega_{beat} = \frac{\omega}{1 - \beta}$.

We will estimate the human audible range to be 20 Hz to 20 kHz. The frequency of the note is the frequency of the sine portion since the beat frequency only modulates volume. Thus, we want $\frac{\beta}{1 - \beta} \approx 10^3$, or $\beta = \frac{1000}{1001} = 0.999001$. 
2.12 Orbit Eccentricity

[Spring 2007 Classical (Afternoon), Problem 1; Thornton & Marion 8.3] A particle moves in a circular orbit in a force field given by \( F(r) = -k/r^2 \). Show that, if \( k \) suddenly decreases to half its original value, the particle’s orbit becomes parabolic.

**SOLUTION:**

Recall that the formula for the eccentricity of the orbit for a reduced mass, \( \mu \), in a central potential with corresponding potential energy \( V(r) = -k/r \) (this produces the force \( F(r) = -dV/dr \)), is

\[
\epsilon = \sqrt{1 + \frac{2EL^2}{\mu k^2}}, \tag{2.12.1}
\]

where \( L \) is the particle’s conserved angular momentum. This is not crucial for this problem, but we can find the initial energy of the particle if we knew its angular momentum since a circular orbit implies \( \epsilon = 0 \).

The virial theorem for a \( 1/r \) potential says that the average kinetic and potential energies are related via \( \langle T \rangle = -\langle V \rangle /2 \). For a circular orbit, these energies are not changing and so \( T = \langle T \rangle \) and \( V = \langle V \rangle \). The total energy is thus \( E = T + U = U/2 \).

At the very instant that \( k \) is cut in half, the particle’s potential energy is immediately cut in half as well (since it’s proportional to \( k \)), but its kinetic energy is not immediately affected. With primed variables indicating that these are right after the change, \( U' = U/2 \) and \( T' = T = -U/2 \). Thus, the energy of the particle now becomes \( E' = T' + U' = -U/2 + U/2 = 0 \). The new eccentricity is thus \( \epsilon' = 1 \), which describes a parabolic orbit.

**Aside:** Remember that \( \epsilon = 0 \) is circular, \( 0 < \epsilon < 1 \) is elliptical, \( \epsilon = 1 \) is parabolic, and \( \epsilon > 1 \) is hyperbolic.
2.13 General Relativistic Correction to Kepler

[Fall 2009 Classical (Afternoon), Problem 3] General Relativity introduces a modification to Kepler’s laws. Surprisingly, it turns out that in General Relativity, the problem of a small particle of mass $m$ orbiting a (spherically symmetric nonrotating) star of mass $M$ (assuming $M \gg m$) is equivalent to the Newtonian problem of a nonrelativistic particle of mass $m$ in a potential

$$V(r) = -\frac{GMm}{r} - \frac{b}{r^3},$$

(2.13.1)

where $G$ is Newton’s gravitational constant, and $b$ is another constant. Consider a bound orbit described by polar coordinates $(r(t), \phi(t))$ as a function of time $t$ and let $L = m|\mathbf{r} \times \mathbf{v}|$ be the magnitude of the total angular momentum.

(a) For a circular orbit, $r(t) = R =$ constant, express $R$ as a function of $L$, and find the period (also as a function of $L$). If you found more than one solution, which one is stable?

(b) Now we assume a near-circular solution, $r(t) = R + \delta(t)$, where $\delta(t) \ll R$ is a small correction. Find an (approximate) expression for $\delta(t)$.

(c) For the near-circular orbit of part (b) above, find an expression for $\phi(t)$ (neglecting corrections that are second-order in $\delta$).

Note: The effective value of $b$ in General Relativity is $b = \frac{GM}{mc^2}L^2$, where $c$ is the speed of light. (It is a function of angular momentum, but don’t worry about that!)

SOLUTION:

(a) $V_{\text{eff}}(r) = \frac{L^2}{2mr^2} - \frac{GMr}{r} - \frac{b}{r^3}$ and circular orbit occurs at extrema of $V_{\text{eff}}$.

$$V'_{\text{eff}}(r) = \frac{GMm}{r^4} \left( r^2 - \frac{L^2}{GMm^2} r + \frac{3b}{GMm} \right),$$

(2.13.2)

whose solutions are

$$r_{\pm} = \frac{L^2}{2GMm^2} \left( 1 \pm \sqrt{1 - \frac{12GMm^3b}{L^4}} \right).$$

(2.13.3)

The stable solution will be the one that is a local minimum of $V_{\text{eff}}$ and the unstable one will be a local maximum. In a moment, we will compute the second derivatives of $V_{\text{eff}}$ evaluated at $r_{\pm}$ to check the nature of each extremum, since we will need it for part (b) anyway. However, we can easily see that $r_+$ is stable and $r_-$ is unstable: at small $r$, the $r^{-3}$ term wins and drives $V_{\text{eff}}$ to $-\infty$ instead of $+\infty$, which is where the $r^{-2}$ term used to drive $V_{\text{eff}}$. A quick drawing should convince you that $r_-$ is a local max and $r_+$ is a local min.

The second derivative is

$$V''_{\text{eff}}(r) = \frac{2GMm}{r^5} \left( r^2 - \frac{3L^2}{2GMm^2} r + \frac{6b}{GMm} \right).$$

(2.13.4)

Evaluating at $r_{\pm}$ yields

$$V''_{\text{eff}}(r_{\pm}) = \frac{2GMm}{r_{\pm}^4} \left( r_{\pm} - \frac{L^2}{2GMm^2} \right) = \pm \frac{L^2}{mr_{\pm}^4} \sqrt{1 - \frac{12GMm^3b}{L^4}}.$$ 

(2.13.5)
The extremum is a local minimum if the second derivative is positive, which occurs only for 
$r_+$. Thus, $r_+$ is stable whereas $r_-$ is unstable.

From now on, set $R = r_+$. $L = mR^2\dot{\phi}$ and so $\dot{\phi} = L/mR^2$. Then, the period is

$$T = \frac{2\pi}{\dot{\phi}} = \frac{2\pi m R^2}{L}.$$  \hfill (2.13.6)

It’s at this point that one usually substitutes $L = m\sqrt{GM}$ and gets the usual Kepler’s third law. However, this last relation no longer holds.

(b) Let us Taylor expand $V_{\text{eff}}$ about $R = r_+$ and recall that $V_{\text{eff}}'(R) = 0$: 

$$V_{\text{eff}}(r) \approx V_{\text{eff}}(R) + \frac{1}{2}V_{\text{eff}}''(R) \delta^2.$$  \hfill (2.13.7)

This is simple harmonic with frequency

$$\omega = \sqrt{V_{\text{eff}}''(R)/m},$$  \hfill (2.13.8)

where $V_{\text{eff}}''(R)$ is the positive solution in (2.13.5).

Thus, the approximate solution to $\delta$ is

$$\delta(t) = \Delta \sin(\omega t + \varphi),$$  \hfill (2.13.9)

where $\Delta$ is the amplitude, $\varphi$ is some phase, and $\omega$ is given in (2.13.8).

(c) The angular momentum is $L = mr^2\dot{\phi}$ and so

$$\dot{\phi} = \frac{L}{mR^2(1 + \delta/R)^2} \approx \frac{L}{mR^2} - \frac{2L\delta}{mR^3}.$$  \hfill (2.13.10)

Substituting in for $\delta(t)$ and integrating over time yields

$$\phi(t) \approx \text{const.} + \frac{Lt}{mR^2} + \frac{2L\Delta}{mR^3\omega} \cos(\omega t + \varphi).$$  \hfill (2.13.11)
2.14 Gravitational Effect of Dust

[Goldstein, Poole & Safko 3.20] A uniform distribution of dust in the solar system adds to the gravitational attraction of the Sun on a planet an addition force \( \mathbf{F} = -mC\hat{r} \), where \( m \) is the mass of the planet and \( C \) is a constant proportional to the gravitational constant and the density of the dust. This additional force is very small compared to the direct Sun-planet gravitational force.

(a) Calculate the period for a circular orbit of radius \( r_0 \) in this combined field.

(b) Calculate the period of radial oscillations for perturbations from this circular orbit.

**SOLUTION:**

(a) The Hamiltonian for a central potential is

\[
H = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) + V(r) = \frac{1}{2} m \dot{r}^2 + V_{\text{eff}}(r),
\]

where the effective potential is given by

\[
V_{\text{eff}}(r) = \frac{L^2}{2mr^2} + V(r),
\]

and \( L = mr^2 \dot{\theta} \) is the conserved angular momentum.

By energy conservation, \( \dot{H} = 0 \), which, after some algebra gives

\[
m \ddot{r} = \frac{L^2}{mr^3} - V'(r),
\]

which, for circular orbits (\( r = r_0 \) is constant) reads

\[
\frac{L^2}{mr_0^3} - V'(r_0) = 0.
\]

Therefore, the angular speed is

\[
\omega = \dot{\theta} = \frac{L}{mr_0^2} = \sqrt{\frac{V'(r_0)}{mr_0}},
\]

Now, write the potential energy as \( V = V_0 + \Delta \), where \( V_0 \) is the usual Newtonian gravitational potential energy and \( \Delta \) is the dust perturbation. Also, define \( \delta \equiv \Delta'/2V_0' \). Then,

\[
\omega = \sqrt{\frac{V_0'(r_0)}{mr_0}} \sqrt{1 + 2\delta} \approx \omega_0 (1 + \delta),
\]

where \( \omega_0 \) is the angular speed without the dust perturbation. Then, the period is given by

\[
\tau = \frac{2\pi}{\omega} \approx \frac{2\pi}{\omega_0} (1 - \delta) = \tau_0 (1 - \delta).
\]

Now, let us calculate \( \omega_0 \) and \( \delta \) explicitly for \( V_0(r) = -k/r \) and \( \Delta = mC r^2 \), which gives the dust force. Then,

\[
\tau_0 = \frac{2\pi}{\sqrt{V_0'/mr_0}} = 2\pi r_0^{3/2} \sqrt{m/k}.
\]
For $\delta$, we find

$$\delta = \frac{\Delta'}{V_0} = \frac{mC r_0^3}{2k} = \frac{C \tau_0^2}{8\pi^2}. \quad (2.14.9)$$

Therefore, the period with the dust perturbation is

$$\tau = \tau_0 \left(1 - \frac{C \tau_0^2}{8\pi^2}\right). \quad (2.14.10)$$

with $\tau_0$ given in (2.14.8).

(b) We set $r = r_0 + x$ and expand (2.14.3):

$$m\ddot{x} = \frac{L^2}{2mr_0^2} \left(1 - \frac{3x}{r_0}\right) - V'(r_0) - xV''(r_0),$$

$$= - \left(\frac{3L^2}{2mr_0^4} - \frac{2k}{r_0^3} + mC\right) x$$

$$= - \left[\frac{3}{r_0} \left(\frac{L^2}{2mr_0^2} - V'(r_0)\right) + \frac{k}{r_0^3} + 4mC\right] x$$

$$= - \left(\frac{k}{r_0^3} + 4mC\right) x, \quad (2.14.11)$$

where we used (2.14.4) to get the second and last lines.

Therefore, the period of oscillations is

$$\tau_{osc} = 2\pi \left(\frac{k}{mr_0^3} + 4C\right)^{-1/2} = \tau_0 \left(1 - \frac{C \tau_0^2}{2\pi^2}\right). \quad (2.14.12)$$

Note that $\tau$ and $\tau_{osc}$ are comparable (since $C$ is small.)
2.15 Falling Through Tunnels in the Earth

Imagine that two straight, small-radius tunnels are drilled through the center of the earth. Tunnel 1 extends along the rotation axis and tunnel 2 begins and ends on the equator. Assuming that the earth has uniform density and is rotating with frequency $\Omega$, how much time does it take for objects dropped into each tunnel to return? Ignore friction and assume the objects start at rest. Is it equally necessary for the two tunnels to be frictionless?

**SOLUTION:**

What is the gravitational force felt by the falling object when it is a radial distance $r$ away from the center of the earth? The gravitational version of Gauss’ law reads $\nabla \cdot \mathbf{g} = -\rho/\kappa$ where $\mathbf{g}$ is the gravitational field, $\rho$ is the mass density and $\kappa$ is the gravitational constant and is related to Newton’s constant by $\kappa = 1/4\pi G$ (Gravity $\leftrightarrow$ E&M: $G \leftrightarrow k$ and $\kappa \leftrightarrow \epsilon_0$.) Note the negative sign in Gauss’ law relative to E&M: in E&M like charges repel, whereas in gravity, masses (there are no unlike masses) always attract.

Assuming that the tunnel is a sufficiently small perturbation on the mass density, the gravitational field is still spherically symmetric and we can write $\mathbf{g} = g \hat{r}$, where $g$ depends only on $r$. Thus, a Gaussian sphere of radius $r$ centered at the center of the earth constitutes a good Gaussian surface. Denote by $D^3$ the solid ball whose boundary is this sphere, $S^2$. Then,

$$4\pi r^2 g = \int_{S^2} \mathbf{g} \cdot d\mathbf{a} \overset{\text{Stokes}}{=} \int_{D^3} \nabla \cdot \mathbf{g} \, dV = -\frac{1}{\kappa} \int_{D^3} \rho \, dV = -\frac{4}{3\kappa} \pi \rho \left[ r^3 - \left( r^3 - R^3 \right) \theta(r-R) \right],$$

where $R$ is the radius of the earth and where $\theta(x)$ is the Heaviside Theta function equal to 0 when $x < 0$ and equal to 1 when $x > 0$ (some will define $\theta(0) = 1/2$, but this is immaterial for our purpose.)

When $r < R$, the $\theta$ function vanishes and we have the standard result

$$g = -\frac{\rho r}{3\kappa}. \quad (2.15.2)$$

The gravitational force felt by the object with mass $m$ is

$$\mathbf{F} = -\frac{m\rho r}{3\kappa} \hat{r}. \quad (2.15.3)$$

**Tunnel 1:** Since the object moves along the axis of rotation, there are no forces due to rotation. Newton’s second law now reads

$$m\ddot{\hat{r}} = \mathbf{F} = -\frac{m\rho r}{3\kappa} \hat{r} \quad \Longrightarrow \quad \ddot{r} + \Omega^2 r = 0 \quad \text{where} \quad \Omega^2 = \rho/3\kappa. \quad (2.15.4)$$

This is just simple harmonic motion. We need the period, which is

$$T_1 = \frac{2\pi}{\Omega} = \frac{2\pi}{\sqrt{\frac{3\kappa}{\rho}}} = \frac{2\pi}{\sqrt{\frac{3}{4\pi G \rho}}} \quad (2.15.5)$$

Plugging in appropriate numbers gives a period of about 1 hour, 24 minutes and 20 seconds. One also finds that a 100 kg object will have a maximum speed of $\sim 2850$ kph or about Mach 2.3 in air!

**Tunnel 2:** Let $\hat{x}$, $\hat{y}$ and $\hat{z}$ be some inertial set of Cartesian coordinates whose origin is the center of the earth (neglect the fact that the center of the earth is rotating around the sun and thus this reference frame is not quite inertial.) For convenience, let $\hat{z}$ be the axis of rotation of
the earth and let the earth rotate in the counter-clockwise direction relative to the positive $\hat{z}$ axis.

In the frame of the object, the object itself is not accelerating. This frame is non-inertial and thus we need to consider fictitious forces. Newton’s second law now reads

$$0 = -\frac{m\rho r}{3\kappa} \ddot{r} + \underbrace{-m\dot{r}\hat{r}}_{\text{translation}} + \underbrace{-m\omega\hat{z} \times (\omega \hat{z} \times \dot{r} \hat{r})}_{\text{centrifugal}} + \underbrace{-2m\dot{\omega}\hat{z} \times \dot{r} \hat{r}}_{\text{Coriolis}} + \underbrace{-m\dot{\omega}\hat{z} \times \dot{r} \hat{r}}_{\text{azimuthal}}. \quad (2.15.6)$$

Actually, this does not include all of the forces, as we will see. Assuming that the earth rotates steadily ($\dot{\omega} = 0$), the azimuthal piece vanishes. Denote by $\hat{\phi}$, the unit vector curling around the origin in the $x, y$-plane in the counter-clockwise direction. Then,

$$0 = -\frac{m\rho r}{3\kappa} \ddot{r} - m\dot{r}\hat{r} + m\omega^2 r \hat{r} - 2m\omega r \hat{\phi}. \quad (2.15.7)$$

Obviously, the wall of the tunnel will prevent the Coriolis force from actually rotating the object in the $\hat{\phi}$ direction. The wall exerts a “normal force” exactly equal and opposite: $N = 2m\omega r \hat{\phi}$. The friction force exerted on the object is directed opposite to the motion: $F_f = -2\mu m\omega r \hat{r}$, where $\mu$ is the coefficient of kinetic friction between the object and the tunnel wall. Thus, Newton’s second law really reads

$$0 = \ddot{r} + 2\mu\omega r + (\Omega^2 - \omega^2) r,$$

(2.15.8)

where $\Omega$ is defined as in (2.15.4).

If $\mu = 0$, as we are told in the problem, then the period is

$$T_2 = \frac{2\pi}{\sqrt{\Omega^2 - \omega^2}} = 2\pi \left( \frac{\rho}{3\kappa} - \omega^2 \right)^{-1/2}.$$

(2.15.9)

Clearly, only Tunnel 2 is required to be frictionless ($\mu = 0$) because the object will inevitably slide along the tunnel wall. Tunnel 1 could have friction as long we drop the object so that it is initially not touching the tunnel walls.

Plugging in appropriate numbers gives a period of about 1 hour, 24 minutes and 29 seconds, or just 9 seconds longer than in Tunnel 1.

Also, we could equally well solve (2.15.8) exactly since it is just a damped harmonic oscillator. As expected, for reasonable values of $\mu$, this system is underdamped.
2.16 Unwinding String

[Kevin G. and Morin 10.11] A wheel with radius $R$ is placed flat on a table. A massless string with one end attached to the rim of the wheel is wrapped clockwise around the wheel a large number of times. When the string is wrapped completely around the wheel, a point mass $m$ is attached to the free end and glued to the wheel. The wheel is then made to rotate with constant angular speed $\omega$. At some point, the glue on the mass breaks and the mass and string gradually unwind (with the speed of the wheel kept constant at $\omega$ by a motor, if necessary). Show that the length of the unwound string increases at the constant rate $R\omega$, for both the clockwise and counterclockwise directions for $\omega$. [The latter is the tricky one.] [It is not necessary to use this, but if you need it, the general solution to the differential equation $(x - \omega t)x' - 2\omega x + x^2 = 0$ is $x(t) = \omega t \pm \sqrt{(\omega t)^2 + At + B}$, where $A$ and $B$ are arbitrary constants.]

SOLUTION:

I mean to emphasize the fictitious forces aspect of this problem. However, it is possible to solve this using the Lagrangian method. Incidentally, this is where the final comment in the question about the solution to a particular differential equation becomes useful. For good measure, I present that solution as well at the end.

Rotating Reference Frame Method 1:

The case when the wheel is rotating in the clockwise direction is actually straightforward. Right when the glue breaks, the velocity of the mass is $R\omega$ and is tangential to the wheel. It just keeps going because there are no other forces acting on it; the only possible force would be a tension in the string, but there is none because the mass travels the same distance in a given amount of time, $t$, as the length of string that unwinds in the same amount of time, namely $R\omega t$. Thus, the rate at which the string unwinds is $R\omega$.

This is no longer the case when the wheel rotates in the counterclockwise direction. The mass will follow a growing spiral path. Consider the problem in the reference frame rotating with frequency $2\omega$ in the counterclockwise direction. In this frame, the wheel looks like it is rotating with frequency $\omega$ in the clockwise direction. So, everything is the same as in the previous case except that there are fictitious forces. The diagram below is in the rotating frame, in which we claim the path of the mass is exactly the same as it was in the lab frame when the wheel was rotating in the clockwise direction. There are three forces acting on the mass: centrifugal ($F_{\text{cent}} = -m\omega' \times (\omega' \times r)$), Coriolis ($F_{\text{Cor}} = -2m\omega' \times v$), and tension. Here, we are taking $\omega' = 2\omega$, where $\omega$ is the counterclockwise rotation of the wheel. This claim would be true if it so happens that the vertical component of $F_{\text{cent}}$ is exactly cancelled by $F_{\text{Cor}}$. Well, in magnitude, $F_{\text{cent}} = m(2\omega)^2 r = 4m\omega^2 r$ and its vertical component is $F_{\text{cent,vert}} = F_{\text{cent}} \frac{R}{r} = 4m\omega^2 R$, whereas $F_{\text{Cor}} = 2m(2\omega)R\omega = 4m\omega^2 R$ and they do in fact exactly cancel!

By the way, this procedure allows us to determine the tension, which is independent of reference frame, although, obviously, it’s rather hard to determine it in any other reference frame except the one we used above. It is equal to the horizontal component of $F_{\text{cent}}$, which is $T = 4m\omega^2 r \cdot \frac{R\omega}{r} = 4mR\omega^3 t$.

![Figure 2.14: Picture in the reference frame rotating counterclockwise at $2\omega$.](image)
Rotating Reference Frame Method 2:
This method is not so obvious. If you rotate to the rest frame of the wheel, in either case, the path of the mass is a growing spiral like in the diagram below.

![Path of the mass in the rest frame of the wheel](image)

Figure 2.15: Path of the mass in the rest frame of the wheel (either cw or ccw).

Note that the velocity of the mass is perpendicular to the string and thus the tension cannot change the speed (magnitude). Of course, $\mathbf{F}_{\text{Cor}}$ can never change the speed since it is always perpendicular to it. Therefore, the only thing that matters insofar as determining the velocity $\mathbf{v}$ as a function of displacement $\mathbf{r}$ is $\mathbf{F}_{\text{cent}}$, which happens to be the same in both cases since $\mathbf{F}_{\text{cent}}$ is unchanged by flipping $\omega$. It follows that the two cases unwind at the same rate, which is $R\omega$, by the simple argument in the lab frame for the clockwise case (beginning of Method 1).

Lagrangian Method:
The diagram below shows a generic situation, where the wheel is rotating with frequency $\omega$ (as drawn, ccw corresponds to positive $\omega$) and the angle at which the string loses contact is $\theta$, and the length of the free string is $\ell$. Obviously, the variables $\ell$ and $\theta$ are related by a constraint.

![Unwinding string](image)

Figure 2.16: Unwinding string.

You may want to take another look at Problem 1.1 since the position of the mass here is almost exactly the same with $\ell$ replacing $\ell$:

$$(x, y) = (R \cos \theta + \ell \sin \theta, R \sin \theta - \ell \cos \theta).$$

(2.16.1)

The unconstrained Lagrangian is $L' = \frac{1}{2}m(x^2 + y^2)$. The constraint is $\ell - R(\theta - \omega t) = 0$. Thus, after some algebra, one finds the total Lagrangian:

$$L = \frac{1}{2}m[(\ell - R\dot{\theta})^2 + \ell^2 \dot{\theta}^2] + \lambda[\ell - R(\theta - \omega t)].$$

(2.16.2)
The $\theta$ and $\ell$ E-L EoM are, respectively,

\begin{align}
-mR\ddot{\ell} + m(\ell^2 + R^2)\dot{\theta} + 2m\ell\dot{\ell}\dot{\theta} + \lambda R &= 0, \\
m(\ddot{\ell} - R\dot{\theta}) - m\ell\dot{\theta}^2 - \lambda &= 0.
\end{align}

(2.16.3)

The constraint equation implies $\dot{\ell} = R(\dot{\theta} - \omega)$ and $\dot{\theta} = R\ddot{\theta}$. Plugging this into the EoM for $\ell$ and solving for $\lambda$ gives $\lambda = -mR(\theta - \omega t)\ddot{\theta}^2$. Plugging all of these back into the EoM for $\theta$ yields a differential equation for $\theta$ identical to the one given at the end of the problem: $(\theta - \omega t)\dot{\theta} - 2\omega\theta + \theta^2 = 0$. The choice of initial condition $\theta(0) = 0$ implies $B = 0$ in the general solution. Then, for the derivative, $\dot{\theta}$, to be sensible at $t = 0$ implies $A = 0$, which leaves us with just $\theta(t) = \omega t \pm \omega t = 0$ or $2\omega t$. The simple argument in the rest frame for the clockwise case shows that $\theta_{cw} = 0$ is the solution. As already mentioned, that is not a viable solution in the ccw case, so $\theta_{cw} = 2\omega t$. Fortuitously, $\dot{\theta} = 2\omega$, when the wheel is rotating at $\omega$, means that the string is unwinding at the rate $R(2\omega - \omega) = R\omega$. 
2.17 Coriolis Force and Bernoulli’s Equation

[Fall 2008 Classical (Afternoon), Problem 3 (augmented)]

(a) A mass \( m \) is dropped from rest at the equator, from an initial height \( H \). Ignore air resistance in this problem.

(i) In what direction is the projectile deflected by the Coriolis force?

(ii) Determine the distance of the horizontal deflection when the mass hits the ground (from the point it would have hit with no deflection), in terms of the initial height \( H \), and the earth’s rotation \( \omega \).

(iii) Evaluate the deflection for \( H = 1.5 \) km.

(b) A water tower of cross-sectional area \( A_1 \) and height \( H \) is filled with water. The top is open to atmospheric pressure. At the bottom is a spigot of area \( A_2 \ll A_1 \).

(i) Find the speed of the water, \( v \), leaving the spigot as a function of the height of the water, \( h \), in the tank.

(ii) Assuming \( h = H \) when the spigot is opened at \( t = 0 \), find the total time it takes to drain the tank.

(iii) Before opening the spigot in part (b), cover the top of the reservoir except for a tube of negligible area submerged down to \( H/2 \). Now, open the spigot. Describe what happens and calculate how long it takes to drain the reservoir.

SOLUTION:

(a) (i) Set up coordinates and local compass directions at a point on the equator as seen in the bird’s eye view below.

Figure 2.18: Local coordinates and compass directions on the equator.

\[ \mathbf{F}_{\text{Cor}} = -2m\omega \times \mathbf{v} = -2m\omega (-v) \mathbf{\hat{z}} \times \mathbf{\hat{x}} = 2m\omega v \mathbf{\hat{y}}. \] Thus, the mass is deflected [eastward]. We neglected the additional Coriolis force due to the gained eastward speed and the centrifugal force since both are \( O(\omega^2) \).

(ii) The ball is dropped at \( t = 0 \) (at rest), \( v = gt \). Thus, \( \mathbf{F}_{\text{Cor}} = 2m\omega gt \mathbf{\hat{y}} \), or \( \mathbf{a}_h = 2\omega gt \mathbf{\hat{y}} \), where \( \mathbf{a}_h \) is the horizontal acceleration. Integrating twice and setting the initial
2.17. CORIOLIS FORCE AND BERNOULLI’S EQUATION

horizontal speed and initial horizontal position equal to 0 gives a horizontal deflection as a function of time of

\[ x(t) = \int_0^t dt' \int_0^{t'} dt'' 2 \omega gt'' = \frac{1}{3} \omega gt^3. \]  

(2.17.1)

It takes a time \( t = \sqrt{2H/g} \) to hit the ground and so the deflection is

\[ x = \omega \sqrt{8H^3/9g}. \]  

(2.17.2)

(iii) \( \omega = 2\pi \text{ rad/24 hrs} = 7 \times 10^{-5} \text{ rad/s} \). Plugging this in gives \( x = 1.22 \text{ m} \).

(b) (i) Denote the surface of the water in the reservoir by 1 and the spigot by 2. Bernoulli’s relation, equating energy densities at the top and bottom, reads

\[ P_1 + \frac{1}{2} \rho v_1^2 + \rho gh = P_2 + \frac{1}{2} \rho v_2^2. \]  

(2.17.3)

Both the top and the spigot exit are at equilibrium with the atmosphere and thus \( P_1 = P_2 = P_{\text{atm}} \). Equating flows gives \( A_1 v_1 = A_2 v_2 \). Therefore, we get

\[ v_2 = \sqrt{\frac{2gh}{1 - (A_2/A_1)^2}}. \]  

(2.17.4)

(ii) Since \( h \) is decreasing, \( v_1 = -\dot{h} \). Thus,

\[ \dot{h} = -\frac{A_2}{A_1} v_2 = -\frac{A_2}{A_1} \sqrt{\frac{2g}{1 - (A_2/A_1)^2}} \sqrt{h}. \]  

(2.17.5)

Integrating in time from 0 to \( t \) and in \( h \) from \( H \) to 0 gives

\[ t = \frac{A_1}{A_2} \sqrt{1 - \left(\frac{A_2}{A_1}\right)^2 \frac{2H}{g}}. \]  

(2.17.6)

(iii) **Stage 1**: the reservoir water level (not in the tube) cannot decrease since air cannot get inside the reservoir. Initially, only the water level in the tube decreases until it reaches the bottom of the tube. Since the tube has negligible area, we will neglect the time for this to happen: \( t_1 \approx 0 \).

**Stage 2**: Now, the pressure at \( H/2 \) is \( P_{\text{atm}} \) since it is at equilibrium with the atmosphere. The pressure there never changes even as the water level in the reservoir decreases. Therefore, the flow rate through the spigot will be constant until the reservoir water level reaches the bottom of the tube.

Let 1 not stand for the level \( H/2 \). Then, \( P_1 = P_2 = P_{\text{atm}} \), and so \( \frac{1}{2} \rho \left(\frac{A_1}{A_1}\right)^2 v_2^2 + \frac{1}{2} \rho gH = \frac{1}{2} \rho v_2^2 \). Thus, \( v_2 = \sqrt{\frac{gH}{1 - (A_2/A_1)^2}} \) and the flow rate \( A_2 v_2 \) is constant. It takes time \( t_2 = A_1 \frac{H}{2} / A_2 v_2 \) to drain. Thus,

\[ t_2 = \frac{A_1}{2A_2} \sqrt{1 - \left(\frac{A_2}{A_1}\right)^2 \frac{H}{g}}. \]  

(2.17.7)
Stage 3: Now, the story is the same as part (b), but starting at $H/2$ rather than $H$. Thus,

$$t_3 = \frac{A_1}{A_2} \sqrt{1 - \left(\frac{A_2}{A_1}\right)^2 \sqrt{\frac{H}{g}}}.$$  \hspace{1cm} (2.17.8)

All together, \[ t = t_1 + t_2 + t_3 = \frac{3A_1}{2A_2} \sqrt{1 - \left(\frac{A_2}{A_1}\right)^2 \sqrt{\frac{H}{g}}} \]
2.18 Siphon and Mariotte’s Bottle

[Kevin G.] Below (left) is a diagram of a basic siphon from wikipedia. Assume that the cross section of the tube is much smaller than the dimensions of the water reservoir that the siphon is draining.

(a) One way to get a siphon started is to “flood” it initially: fill it with water and cover end C before submerging end A and removing the cover on C. Explain how the siphon is able to drain water out of the reservoir.

(b) What is the speed of water flow out in terms of \( h_c \)? What is the maximum speed of smooth water flow out in terms of \( h_b \)? What is the maximum height \( h_b \)?

(c) In a real system such as the bottle on the right, the water level in the bottle will decrease as it is drained. The flow rate is fastest in the beginning and gradually decays. To keep the flow rate constant, form a Mariotte’s Bottle by inserting a second vertical tube open to the outside. For simplicity submerge the siphon and the second tube to the same depth (essentially as deep as possible). Explain how this keeps the flow rate constant.

Figure 2.19: Siphon and Mariotte’s Bottle.

SOLUTION:

(a) The pressure at A is higher than at C, which is at atmospheric pressure, \( P_0 \). Hence, water will flow through the tube from A to C.

Another way to see this is that, at the beginning, before the cover at C is removed, the water in column BC is heavier than the one in BA. This will cause the BC column to fall when the cover is removed. Assuming the water column doesn’t break (cavitation), atmospheric pressure on the water surface in the reservoir will continue to push water through the tube.

(b) Since the siphon cross section is much smaller than the reservoir dimensions, we can treat the reservoir as essentially infinite: the speed at which the water level in the reservoir is decreasing is negligible compared to the flow speed through the tube. Set the origin of vertical height at the water level in the reservoir. Then, the constant in Bernoulli’s equation is \( \frac{1}{2} \rho(0)^2 + \rho g(0) + P_b = P_0 \). Bernoulli states that this expression is the same at all points along the water flow. At C, this reads \( \frac{1}{2} \rho v_c^2 - \rho gh_c + P_0 = P_0 \), or \( v_c = \sqrt{2gh_c} \).

At B, this reads \( \frac{1}{2} \rho v_b^2 + \rho gh_b + P_b = P_0 \). We can drive \( P_b \) down to 0, but no further, or else cavitation occurs (bubble formation) and the flow will no longer be smooth. Setting \( P_b = 0 \) gives

\[
 v_{\text{max}} = \sqrt{2 \left( \frac{P_0}{\rho} - gh_b \right)} .
\]

(2.18.1)

Requiring the above speed to be \( \geq 0 \) yields the maximum height: \( h_{b,\text{max}} = \frac{P_0}{\rho g} \).
(c) If the siphon is started as in part (a), then water will start to flow out. Initially, only the water level inside the second tube decreases until it reaches the bottom of the tube and air starts to enter the bottle via the second tube. At this point, the pressure at the submerged ends of the tubes is atmospheric pressure, $P_0$, and it remains constant until the water level in the bottle reaches the ends of the tubes. Since the intake pressure is constant the whole time, as opposed to decreasing with the decreasing water level, the water flows out at a constant rate.
2.19 Buoyancy

(Spring 2005 Classical (Morning), Problem 2) A solid wooden disk with radius $a$, thickness $t$ and mass density $\rho_w$ floats in a deep cylindrical container of radius $R$ partially filled with fluid with mass density $\rho_f$. The disk is pushed down almost to submersion and released, whereupon it bobs up and down (always remaining upright) without damping. Determine the angular frequency $\omega$ of the vertical oscillations of the disk. Do not assume $a \ll R$.

![Diagram of a wooden disk floating in a fluid container](image)

Figure 2.20: Spring 2005 Classical (Morning), Problem 2

**SOLUTION:**

Let $h_0$ be the height of the water level in the cylinder without the wooden disk. Let $y$ denote how far below $h_0$ the bottom of the wooden disk goes and $x$ denote how far above $h_0$ the water level rises. Conservation of fluid volume implies $\pi R^2 h_0 = \pi R^2 (h_0 + x) - \pi a^2 (x + y)$ or

$$x = a^2 y \frac{R^2 - a^2}{R^2}.$$  \hspace{1cm} (2.19.1)

The gravitational force is $F_g = \pi a^2 t \rho_w g$ and the buoyancy is $F_b = \pi a^2 (x + y) \rho_f g$. Hence, the net restoring force is $\sum F = \pi a^2 t \rho_w g \left[1 - \frac{\rho_f}{\rho_w} \frac{x + y}{t}\right] = m_w \left[1 - \frac{\rho_f}{\rho_w} \frac{R^2 - a^2}{R^2} \frac{y}{t}\right]$, where $m_w$ is the mass of the wooden disk. Setting this equal to $m_w \ddot{y}$ gives

$$\ddot{y} + \frac{g \rho_f}{t \rho_w} \frac{R^2}{R^2 - a^2} y = g,$$  \hspace{1cm} (2.19.2)

which is simply a displaced SHO equation. That is, you can figure out a translation $y \to y + y_0$ that will get rid of the $g$ on the RHS and leaves the term linear in $y$ invariant. Hence, the frequency is given by

$$\omega^2 = \frac{g \rho_f}{t \rho_w} \frac{R^2}{R^2 - a^2}.$$  \hspace{1cm} (2.19.3)
2.20 Viscous Flow through a Tube

Spring 2000 Classical (Morning), Problem 6 A viscous fluid is moving at a constant rate through a cylindrical pipe of radius $R$ and length $L$ under a pressure difference of $p_1 - p_2$. Show that the velocity profile of the liquid as a function of the distance from the axis is given by $v(r) = \frac{p_1 - p_2}{4\eta L} (R^2 - r^2)$, where $\eta$ is the viscosity coefficient defined by the relation $F_{\text{visc}}/A = -\eta (dv/dr)$ between the sheer force per unit area along the pipe and the velocity gradient in the radial direction. Neglect any end effects. Assume that the fluid velocity at the surface of the pipe vanishes.

![Figure 2.21: Spring 2000 Classical (Morning), Problem 6](image)

SOLUTION:

Consider a solid cylinder of fluid of length $L$ and radius $r$. It experiences a force on each of its ends given by the corresponding pressure multiplied by the cross-sectional area. Hence, the force due to pressure experienced by this cylinder of fluid is $F_p = (p_1 - p_2)\pi r^2$. If the flow of the fluid is to be constant, then this bit of fluid cannot accelerate and hence the viscous drag must exactly cancel this force due to pressure.

The viscous force tends to sheer the cylinder turning it into a volume of revolution of a long chevron. The sheer force acts along the surface of the cylinder, so the area in the viscous flow is $2\pi r L$ (not the cross-sectional area): $F_v = -2\pi r L \eta (dv/dr)$. Setting $F_v = F_p$ yields

$$\int_{v(r)}^{0} dv = - \int_{r}^{R} \frac{(p_1 - p_2)r}{2\eta L} dr,$$

where we have used the fact (or assumption) that the speed of the fluid right at the surface of the cylinder vanishes, and we have separated variables and integrated. Solving for $v(r)$ yields

$$v(r) = \frac{(p_1 - p_2)(R^2 - r^2)}{4\eta L}.$$
2.21 Air Resistance and Projectile Motion

An object of mass $m$, with cross-sectional area $A$, is shot up ballistically with an initial speed, $v_0$, and at an angle $\theta$ relative to flat ground.

(a) Neglecting air resistance, what is the horizontal range of the projectile?

(b) Use a simple model (e.g. stationary air molecules impacted by a moving object) to derive an equation giving the force on the projectile from air resistance.

(c) At relatively low speeds, air drag is better modeled by Stoke’s law for viscous flow. Suppose the object is a sphere of radius $R$. Then $F_{\text{air}} \approx -6\pi \eta R v$, where $\eta$ is the viscosity of air. Determine the position of the projectile as a function of time and verify the small air resistance limit. Calculate the first order correction to the horizontal range.

SOLUTION:

(a) The initial vertical speed is $v_0 \sin \theta$. At the top, the vertical speed is 0. Using kinematics, $v_f = v_i + at$, with $v_i = v_0 \sin \theta$, $v_f = 0$ and $a = -g$ yields $t = \frac{v_0}{g} \sin \theta$. The time to hit the ground is twice this. The horizontal distance travelled in this time is

$$R = (v_0 \cos \theta) \left( \frac{2v_0}{g} \sin \theta \right) = \frac{v_0^2}{g} \sin 2\theta$$

(2.21.1)

(b) Before the collision between the projectile and the air molecules, the speed of the air molecules is 0 and speed of projectile is $v$. Afterwards, suppose speed of air molecules is $v$. The mass of air in volume $\Delta V$ is $\rho \Delta V$. The momentum imparted on the air molecules is $p = (\rho A \ell) v = \rho A v \ell$. The force on air is $F = \dot{p} = \rho A v \ell = \rho A v^2$. In practice, the real answer is between 1/2 to 2 of this. The force on the projectile is just the opposite: $F_{\text{air}} = -\rho A v v$.

(c) Let $\tau = m / 6\pi \eta R$, so that $F_{\text{air}} = -\frac{m}{\tau} v$. Newton’s third law now reads

$$\ddot{x} = \frac{1}{\tau} (\dot{y} + \dot{\tau}) = -\frac{1}{\tau} \dot{x} - (g + \frac{1}{\tau} \dot{y}) \dot{y}. \quad (2.21.2)$$

With a multiplying factor, $\frac{d}{dt}(e^{t/\tau}) = e^{t/\tau}(\ddot{x} + \frac{1}{\tau} \dot{x}) = 0$. Hence, $\dot{x} = Ce^{-t/\tau}$. $\dot{x}(0) = v_0 \cos \theta$ implies $\dot{x} = v_0 \cos \theta e^{-t/\tau}$ and thus, $x = -v_0 \tau \cos \theta e^{-t/\tau} + C'$. $x(0) = 0$ implies $x(t) = v_0 \tau \cos \theta (1 - e^{-t/\tau})$. Similarly, $\frac{d}{dt}(e^{t/\tau}) \dot{y} = e^{t/\tau}(\ddot{y} + \frac{1}{\tau} \dot{y}) = -e^{t/\tau} y = \frac{d}{dt}(-g \tau e^{t/\tau})$. Hence, $\dot{y} = -g \tau + Ce^{-t/\tau}$. $\dot{y}(0) = v_0 \sin \theta$ implies $\dot{y} = -g \tau + (v_0 \sin \theta + g \tau) e^{-t/\tau}$ and thus, $y = -g \tau t - (v_0 \sin \theta + g \tau) e^{-t/\tau} + C'$. $y(0) = 0$ implies $C' = (v_0 \sin \theta + g \tau) \tau$ and thus, $y(t) = -g \tau t + (v_0 \sin \theta + g \tau) \tau (1 - e^{-t/\tau})$.

Weak air resistance is described by large $\tau$, or $\bar{t} \equiv t/\tau \ll 1$. Then $e^{-\bar{t}} \approx 1 - \bar{t} + \frac{1}{2} \bar{t}^2$. Thus,

$$x(t) \approx (v_0 \tau \cos \theta) \bar{t}[1 - \frac{1}{2} \bar{t}] \quad \text{and} \quad y(t) \approx (v_0 \tau \sin \theta) \bar{t}[1 - \frac{1}{2} \bar{t}] - \frac{1}{3} g \bar{t}^2 [1 - \frac{1}{3} \bar{t}] \quad (2.21.3)$$

To zeroth order, we keep only 1 in the square brackets and we get back the usual no air-resistance position: $x(t) \approx (v_0 \cos \theta) \bar{t}$ and $y(t) \approx (v_0 \sin \theta) \bar{t} - \frac{1}{2} g \bar{t}^2$. 

Set $\bar{t}_0 \equiv 2 \frac{v_0}{g} \sin \theta$, which is the zeroth order time it takes to hit the ground (in units of $\tau$). Define $\bar{y} \equiv y / g \tau^2$. Then,

$$
\bar{y}(\bar{t}) = -\bar{t} + (1 + \frac{\bar{t}_0}{2}) \left[ 1 - e^{-\bar{t}} \right] \\
\approx -\bar{t} + (1 + \frac{\bar{t}_0}{2}) \left[ 1 - (1 - \bar{t} + \frac{\bar{t}^2}{2} - \frac{\bar{t}_0^2}{6} + \frac{\bar{t}_0^4}{24}) \right] \\
= \frac{\bar{t}}{2} \left[ \bar{t}_0 - (1 + \frac{\bar{t}_0}{2}) \bar{t} + \frac{\bar{t}^2}{2} + \frac{\bar{t}_0^2}{12} \right] \\
= (1 + \frac{\bar{t}_0}{2}) \frac{3\bar{t}}{4} \left[ \frac{-\bar{t}^3}{8} + \frac{\bar{t}^2}{2} - \frac{\bar{t}_0^2}{3} + \frac{3\bar{t}_0}{2} (1 + \frac{\bar{t}_0}{2}) \right].
$$

(2.21.4)

We must solve $\bar{y}(\bar{t}) = 0$. Let us just keep up to the quadratic equation, whose solutions are

$$
\bar{t} = \frac{3}{2} \left\{ 1 \pm \sqrt{1 - \left( \frac{2}{3} \right)^2 3\bar{t}_0 (1 + \frac{\bar{t}_0}{2})^{-1}} \right\}^{1/2}. 
$$

(2.21.5)

We must pick the top sign so that $\bar{t} \approx \bar{t}_0$. In fact, $\bar{t} \approx \bar{t}_0 (1 - \frac{\bar{t}_0}{2})$. The zeroth order range $R^{(0)} = (v_0 \tau \cos \theta) \bar{t}_0$ and $\bar{x}(\bar{t}) \equiv x(\bar{t}) / R^{(0)} = \frac{\bar{t}}{\bar{t}_0} (1 - \frac{\bar{t}}{2})$. Evaluating this at $\bar{t} \approx \bar{t}_0 (1 - \frac{\bar{t}_0}{2})$ gives the first order range in units of $R^{(0)}$:

$$
\bar{R} = (1 - \frac{\bar{t}_0}{2}) \left[ 1 - \frac{\bar{t}_0}{2} (1 - \frac{\bar{t}_0}{2}) \right] \approx 1 - \bar{t}_0 + O(\bar{t}_0^2).
$$

(2.21.6)

Hence, $R^{(1)} = R^{(0)} \left( 1 - \frac{2v_0 \sin \theta}{g \tau} \right)$.
Part II

Electricity and Magnetism
Electrostatics

The diagram below (Figure 2.35 in Griffiths) pretty much encapsulates electrostatics.

Figure 3.1: Summary of electrostatic relations.

- Important boundary condition for a sheet of charge: $E_{\text{above}} - E_{\text{below}} = \frac{\rho}{\varepsilon_0} \hat{n}$.
- Pressure on sheet of charge: $P = \frac{1}{2} (E_{\text{above}} + E_{\text{below}}) \sigma$.
- Energy density: $\frac{1}{2} \rho V$ (in charges) and $\frac{1}{2} \varepsilon_0 E^2$ (in field). [Note: the first does not count the energy needed to create the charges, which for point charges is infinite; thus, the second is not used for point charges. For continuous charge distributions, the distinction is usually negligible.]
- Conductor: in equilibrium, (a) $E = 0$ and $\rho = 0$ in the bulk; (b) conductor is an equipotential; (c) $E \perp$ surfaces.
- Self-capacitance: how much charge must be placed on an isolated conductor to raise its potential by one volt, $C_s = Q/V$. [Note: usually $V = 0$ at $\infty$.]
- Mutual-capacitance: two bodies; (1) place $+Q$ on one and $-Q$ on the other; (2) compute the potential difference, $V$, between the two bodies; (3) $C_m = Q/V$.  

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3.1 Modified Electrostatics

[Griffiths 2.49] You have just performed an experiment demonstrating that the 
actual force of interaction between two point charges is
\[ F = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r^2} \left(1 + \frac{r}{\lambda}\right) e^{-r/\lambda} \hat{r}, \]  
(3.1.1)
where $\lambda$ is a new constant of nature (it has dimensions of length, obviously, and is a huge number - say half the radius of the known universe - so that the correction is small). Assuming the principle of superposition still holds, reformulate electrostatics to accommodate your new discovery.

(a) What is the electric field of a charge distribution $\rho$?

(b) Does this electric field admit a scalar potential? If so, what is it? If not, why not?

(c) What is the new version of Gauss’ law?

(d) Draw the new and improved triangle diagram (like Griffiths Fig. 2.35).

**SOLUTION:**

(a) $E_{pt} = F/Q$. By superposition, 
\[ E = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho \hat{\phi}}{\rho^2} \left(1 + \frac{\rho}{\lambda}\right) e^{-\rho/\lambda} d\tau \]
(b) Yes. For a point charge, $E$ is still spherically symmetric and purely radial, so $\nabla \times E = 0$ and thus $E = -\nabla V$ for some potential $V$. Take a purely radial path directly from $\infty$ to $r$:
\[ V_{pt}(r) = -\int_{\infty}^{r} E_{pt} \cdot d\ell = -\frac{q}{4\pi\varepsilon_0} \int_{\infty}^{r} \frac{1}{r^2} \left(1 + \frac{r'}{\lambda}\right) e^{-r'/\lambda} dr' = \frac{q}{4\pi\varepsilon_0} \frac{e^{-r/\lambda}}{r}, \]  
(3.1.2)
which is none other than the Yukawa potential. We can interpret this as the photon gaining an extremely small, but finite, mass, $m_\gamma \sim 1/\lambda$.

For arbitrary charge distributions, we have
\[ V(r) = \frac{1}{4\pi\varepsilon_0} \int_{\mathcal{V}} \frac{\rho(r') e^{-\rho/\lambda}}{\rho} dr'. \]  
(3.1.3)

(c) Let us compute the divergence of the electric field
\[ \nabla_r \cdot E(r) = \frac{1}{4\pi\varepsilon_0} \int_{\mathcal{V}'} \left\{ \rho(r') \left(1 + \frac{\rho}{\lambda}\right) e^{-\rho/\lambda} \nabla_r \cdot \hat{\phi} + \frac{\rho(r') \hat{\phi}}{\rho^2} \cdot \nabla_r \left[ \left(1 + \frac{\rho}{\lambda}\right) e^{-\rho/\lambda} \right] \right\} d\tau' \]
\[ = \frac{1}{4\pi\varepsilon_0} \int_{\mathcal{V}'} \left\{ \rho(r') \left(1 + \frac{\rho}{\lambda}\right) e^{-\rho/\lambda} 4\pi \delta(\hat{\phi}) - \frac{\rho(r') e^{-\rho/\lambda}}{\lambda^2 \rho} \right\} d\tau' \]
\[ = \frac{\rho(r)}{\varepsilon_0} - \frac{V(r)}{\lambda^2}. \]  
(3.1.4)

Note that the second line follows since we can change $\nabla_r$ to $\nabla_\rho$. In integral form, this reads
\[ \int_S \mathbf{E} \cdot d\mathbf{a} + \frac{1}{\lambda^2} \int_{\mathcal{V}} V d\tau = \frac{q_{enc}}{\varepsilon_0}, \]  
(3.1.5)

(d) The triangle diagram of a summary of the above.
3.1. MODIFIED ELECTROSTATICS

Figure 3.2: Yukawa electrostatics triangle diagram.
Chapter 4

Special Techniques

- Poisson’s equation: $\nabla^2 V = -\rho/\epsilon_0$. Laplace’s equation is just when $\rho = 0$.
- Laplace’s equation forbids local extrema in $V$ and its solution is unique.
- Image charge for a plane: opposite charge at the mirror image point.
- Image charge for a sphere of radius $R$: $q$ outside at distance $a$; $q' = -Rq/a$ inside at distance $b = R^2/a$.
- I don’t think the separation of variables section is crucial.
- Dipole: $V(r) = \frac{1}{4\pi\epsilon_0r} \int r' \cos \theta' \rho(r') \, d\tau'$.
- Quadrupole: $V(r) = \frac{1}{4\pi\epsilon_0r^2} \int r'^2 \left( \frac{3}{2} \cos^2 \theta' - \frac{1}{2} \right) \rho(r') \, d\tau'$.
- Multipole: $V(r) = \frac{1}{4\pi\epsilon_0 r^{n+2}} \int r'^n P_n(\cos \theta') \rho(r') \, d\tau'$.
- Dipole moment $p$ electric field: $E(r) = \frac{1}{4\pi\epsilon_0 r^3} [3(p \cdot \hat{r})\hat{r} - p] - \frac{1}{4\pi\epsilon_0} p \delta(r)$. If you want to derive the delta function, see 3.41 and 3.42 of Griffiths (p.156-157).
4.1 Charge between Parallel Conducting Planes

[Fall 2008 Classical (Afternoon), Problem 4; Griffiths 3.44] Two infinite conducting planes are arranged in a parallel configuration, with one in the $x = 0$ plane, and the other in the $x = D$ plane. Both of these conducting planes are grounded. A point charge $q$ is introduced between the two planes, at the position $(x, y, z) = (a, 0, 0)$. We would like to calculate the result potential between the plates. This problem can be solved using the method of images.

(a) Determine the series of image charges that can reproduce the potential between these plates. (Warning, infinite series may be in your near future.)

(b) From these image charges, derive the potential function between the plates.

(c) Determine the surface charge density as a function of radial position, $r = \sqrt{y^2 + z^2}$, on the surface of the $x = 0$ plate.

(d) Compute the integral of the surface charge density, $\sigma$, over the whole $x = 0$ plane. You should reach a complication. Use Green’s reciprocity theorem to resolve this issue. The theorem says that if one problem contains a charge distribution, $\rho_1$, creating a potential $V_1$, and another problem has $\rho_2$ and $V_2$, then $\int \rho_1 V_2 \, d\tau = \int \rho_2 V_1 \, d\tau$, where the integrals are over all of space. [Hint: for distribution 1, use the actual situation; for distribution 2, remove $q$, and set one of the conductors at potential $V_0$.]

SOLUTION:

(a) Place a charge $q$ at $x = 2nD + a$ and $-q$ at $x = 2nD - a$ for $n \in \mathbb{Z}$.

(b) \[
V(x, y, z) = \frac{q}{4\pi\varepsilon_0} \sum_{n \in \mathbb{Z}} \left\{ \frac{1}{\sqrt{(x-(2nD+a))^2+y^2+z^2}} - \frac{1}{\sqrt{(x-(2nD-a))^2+y^2+z^2}} \right\}.
\]

(c) Let the electric field at $x = \epsilon > 0$, a positive small number ($\epsilon/D << 1$), be $E$, and the field at $x = -\epsilon$ be $E'$. Then, $(E - E') \cdot \hat{x} = \sigma/\epsilon_0$, where $\sigma$ is the charge density on the $x = 0$ plane. Since $x = -\epsilon$ is outside the region between the two plates, $E' = 0$. Hence, $\sigma = \epsilon_0 E_x = -\epsilon_0 \left. \frac{\partial V}{\partial x} \right|_{x=0}$. Defining $r^2 = y^2 + z^2$, this gives

\[
\sigma(r) = -\frac{q}{4\pi} \sum_{n \in \mathbb{Z}} \left\{ \frac{2nD+a}{[(2nD+a)^2+r^2]^{3/2}} - \frac{2nD-a}{[(2nD-a)^2+r^2]^{3/2}} \right\}.
\]

Figure 4.1: Fall 2008 Classical (Afternoon), Problem 4
(d) If we are allowed to interchange the summation and integration, then

\[
\int \sigma = -\frac{q}{2\pi} \sum_{n \in \mathbb{Z}} \left\{ \int_{0}^{\infty} \frac{(2nD + a) r \, dr}{[(2nD + a)^2 + r^2]^{3/2}} - \int_{0}^{\infty} \frac{(2nD - a) r \, dr}{[(2nD - a)^2 + r^2]^{3/2}} \right\}
\]

\[
= \frac{q}{2\pi} \sum_{n \in \mathbb{Z}} \{ \text{sgn}(2nD + a) - \text{sgn}(2nD - a) \}. \tag{4.1.2}
\]

This last expression is not well-defined; depending on how we group terms, it can equal any integer! The problem is that, in interchanging the integral and the sum, we have switched two limits: \( r \to \infty \) and \( n \to \infty \).

As hinted in the problem, there is a much simpler way to solve this problem. Let \( Q_{1\ell} \) be the total charge on the left plate \((x = 0)\), \( Q_{1r} \), on the right plate \((x = D)\), and \( Q_{1a} = q_1 \), at \( x = a \). Similarly, define the potentials \( V_{1\ell} = V_{1r} = 0 \) and \( V_{2a} \). Similarly define the same quantities for situation 2, in which case, \( V_{2\ell} = 0, V_{2r} = V_0, V_{2a} = \frac{q}{D} V_0 \) and \( Q_{2a} = 0 \). Thus, we have Green’s reciprocity gives

\[
\int \rho_2 V_1 = V_{1\ell} Q_{2\ell} + V_{1r} Q_{2r} = 0,
\]

\[
\int \rho_1 V_2 = V_{2\ell} Q_{1\ell} + V_{2r} Q_{1r} + V_{2a} Q_{1a} = V_0 Q_{1r} + \frac{q}{D} V_0 q_1. \tag{4.1.3}
\]

Green’s reciprocity implies that these are equal. Thus,

\[
Q_{1r} = -\frac{q}{D} q_1. \tag{4.1.4}
\]

We are asked for \( Q_{1\ell} \), the charge on the \( x = 0 \) plane in the first situation. This we can determine by noting that the total induced charge on the two plates must be \(-q_1\) in order for the potential to vanish at least very far away on either side of the plates. Hence,

\[
Q_{1\ell} = -(1 - \frac{q}{D}) q_1. \tag{4.1.5}
\]
4.2 Charge Inside an Oscillating Conducting Sphere

[Spring 2008 Classical (Morning), Problem 5] A hollow metal sphere of mass \( m \) and radius \( R \) is electrically grounded, and is connected to a spring (spring constant \( k \)) such that it can oscillate horizontally (\( x \)-axis) around \( x = 0 \). If a point charge \( q \) is fixed at \( x = 0 \), show that for \( q < q_c \), the sphere can be considered a simple harmonic oscillator for small amplitude oscillation. Find the oscillation frequency and the critical charge \( q_c \). Ignore all frictions. Assume the sphere slides on the floor without friction. The spring is in its relaxed (equilibrium) position at \( x = 0 \).

Figure 4.2: Spring 2008 Classical (Morning), Problem 5

SOLUTION:

We need the potential over the sphere to vanish. This is one of the mirror charge configurations we can solve. Suppose that the sphere is displaced a distance \( x \) to the right, so that \( q \) is to the left of the center of the sphere. Let us place a mirror charge \( \alpha q \) (for some \( \alpha \)) somewhere to the left of the sphere, a distance \( r \) from the center of the sphere. Let \( r_1 \) be the distance between \( q \) and an arbitrary point on the sphere, and \( r_2 \) between \( \alpha q \) and the same point. The vanishing of the potential on the sphere implies

\[
\frac{1}{r_1} = \frac{-\alpha}{r_2}. \quad (4.2.1)
\]

Choose two specific points on the sphere: the nearest and farthers points to \( \alpha q \). These yield the equations

\[
\frac{1}{R-x} = -\frac{\alpha}{r-R}, \quad \frac{1}{R+x} = -\frac{\alpha}{r+R}, \quad (4.2.2)
\]

which allow us to solve for \( r \) and \( \alpha \):

\[
r = R^2/x, \quad \alpha = -R/x. \quad (4.2.3)
\]

The force felt by \( q \) is

\[
\mathbf{F}_q = \frac{\alpha q^2}{4\pi\epsilon_0 (r-x)^2} \hat{x} = -\frac{q^2 R}{4\pi\epsilon_0} \frac{x}{(R^2-x^2)^2} \approx -\left( \frac{q^2}{4\pi\epsilon_0 R^3} \right) x \hat{x}. \quad (4.2.4)
\]

By Newton’s third law, the sphere feels negative of this force (in addition to the spring, of course). Hence, the force on the sphere is

\[
\mathbf{F}_s = -\left( k - \frac{q^2}{4\pi\epsilon_0 R^3} \right) x \hat{x}, \quad (4.2.5)
\]

and the oscillation frequency is

\[
\omega = \sqrt{\frac{k}{m} - \frac{q^2}{4\pi\epsilon_0 R^3 m}} = \omega_0 \sqrt{1 - (q/q_c)^2}, \quad (4.2.6)
\]

where \( \omega_0 = \sqrt{k/m} \) and the critical charge is

\[
q_c = \sqrt{4\pi\epsilon_0 R^3 k}. \quad (4.2.7)
\]
4.3 Conducting Plane with a Bump

[Jackson 2.10] An infinite plane conductor has a small protrusion in the form of a hemisphere of radius $R$. A point charge is placed directly above the protrusion, at a height $h$ above the plane. The conductor is kept grounded, and the electric field vanishes at infinity.

(a) Find the electrostatic potential.

(b) Find the total charge that is induced on the protrusion.

(c) Now take the limit $h \to \infty$, $Q \to \infty$, keeping $Q/h^2$ fixed, so that the effect of the charge $Q$ is to produce, far away from the protrusion, a uniform electric field of magnitude $E_0 = Q/(2\pi \varepsilon_0 h^2)$. How does the protrusion modify the electrostatic potential?

\[ \Phi = \frac{Q}{4\pi \varepsilon_0} \left( \frac{1}{\sqrt{\rho^2+(z-h)^2}} - \frac{1}{\sqrt{\rho^2+(z+h)^2}} + \frac{R}{h} \left( \frac{1}{\sqrt{\rho^2+(z+R^2/h)^2}} - \frac{1}{\sqrt{\rho^2+(z-R^2/h)^2}} \right) \right) \]  

(4.3.1)

- The total charge induced on the plane and protrusion is $-Q$ since this is the sum of all of the image charges used. We can compute the total charge induced on the planar part and subtract this from $-Q$. The normal vector to the planar part is $\hat{z}$. Thus, we just need to compute $-\partial \Phi / \partial z$ and evaluate at $z = 0$:

\[
-\left. \frac{\partial \Phi}{\partial z} \right|_{z=0} = -\frac{Q}{4\pi \varepsilon_0} \left( \frac{-z + h}{(\rho^2 + (z-h)^2)^{3/2}} + \frac{z + h}{(\rho^2 + (z+h)^2)^{3/2}} - \cdots \right) - \frac{R}{h} \left( \frac{z + R^2/h}{(\rho^2 + (z+R^2/h)^2)^{3/2}} - \frac{z - R^2/h}{(\rho^2 + (z-R^2/h)^2)^{3/2}} \right) \bigg|_{z=0} \\
= \frac{-Qh}{2\pi \varepsilon_0} \left( \frac{1}{(\rho^2 + h^2)^{3/2}} - \left( \frac{R}{h} \right)^3 \frac{1}{(\rho^2 + (R^2/h)^2)^{3/2}} \right) 
\]

(4.3.2)

This is set equal to $\sigma/\varepsilon_0$, which just gets rid of the overall minus sign and the $\varepsilon_0$ in the denominator of (4.3.2). Notice that there is no azimuthal dependence and so the angular...
integral just cancels the $2\pi$ in the denominator. Then, we must integrate over $\rho$:

$$Q_{\text{induced}}^{\text{plane}} = -Qh \int_{R}^{\infty} \frac{\rho \, d\rho}{(\rho^2 + h^2)^{3/2}} + \frac{QR^3}{h^2} \int_{R}^{\infty} \frac{\rho \, d\rho}{(\rho^2 + (R^2/h^2))^{3/2}}$$

$$= \frac{Qh}{\sqrt{\rho^2 + h^2}} \bigg|_{R}^{\infty} - \frac{QR^2}{h^2} \frac{1}{\sqrt{\rho^2 + (R^2/h^2)}} \bigg|_{R}^{\infty}$$

$$= -\left( \frac{1 - (R/h)^2}{\sqrt{1 + (R/h)^2}} \right) Q \quad (4.3.3)$$

Therefore, the total charge induced on the protrusion is

$$Q_{\text{induced}}^{\text{protrusion}} = -\left[ 1 - \left( \frac{1 - (R/h)^2}{\sqrt{1 + (R/h)^2}} \right) \right] Q \quad (4.3.4)$$

(c) As $h \to \infty$, the two image charges near the protrusion tend towards the origin and thus approximate a perfect dipole of dipole moment

$$P_{\text{protrusion}} = \left( \frac{QR}{h} \right) \left( \frac{2R^2}{h^2} \right) (-\hat{z}) = \left( \frac{-2Q\hat{z}}{h^2} \right) R^3 \quad (4.3.5)$$

Thus, the uniform field $E_0 = -E_0\hat{z}$ where $E_0 = Q/\kappa \varepsilon_0 h^3$ will be modified by the dipole field $\frac{1}{4\pi\varepsilon_0} \frac{P_{\text{protrusion}} \cdot \mathbf{r}}{r^3}$ with $P_{\text{protrusion}}$ given in (4.3.5).
4.4 Earnshaw’s Theorem

[Griffiths 3.2] Consider the cubical arrangement of fixed positive charges $q$ at the corners of a cube. Could you suspend a positive charge at the center in a stable manner (not counting quantum tunneling; in other words, is there any potential barrier to tunnel through?)

**SOLUTION:**

No, there cannot be any potential barrier to tunnel through for such a thing would imply that the potential for this system, $V(r)$, has at least a local minimum at the center. Laplace’s equation forbids this: potentials can only have extrema at the boundaries of the appropriate region (here, the boundary is at infinity) or in the vicinity of charges.

If you plot the potential, for example, on a central slice (e.g. the $xy$ plane), then you will find that the apparent local minimum at the center is in fact a saddle point and it becomes clear that the particle will simply roll down through the centers of the faces. This also becomes apparent if you draw the electric field along this slice. These are shown below.

![Figure 4.3: Field and potential point charges fixed at the corners of a cube.](image)
4.5 Dodecahedron with One Charged Face

[Kevin G.] A dodecahedron has twelve regular-pentagonal faces. All the faces are grounded, but one, which sits at a potential \( V \). What is the potential at the very center of the dodecahedron (i.e. the center of the sphere that circumscribes the dodecahedron.)

**SOLUTION:**

Call the potential at the center, \( V_c \). By symmetry, \( V_c \) does not depend on which face sits at potential \( V \). In particular, by superposition, \( 12V_c \) would be the potential at the center if all faces sat at potential \( V \).

Now, consider Laplace’s equation in the region inside the dodecahedron when all the faces are at potential \( V \). The value of the potential, \( \phi \), is \( V \) at the boundary of this region and satisfies Laplace’s equation, \( \nabla^2 \phi = 0 \), everywhere in the bulk. One obvious solution is the constant \( \phi = V \), which must be the solution since the solution to Laplace’s equation, with fully specified boundary conditions, is unique. Hence, the potential at the center in this case is simply \( V \).

The previous two paragraphs imply \( V_c = V/12 \).
4.6 Charge in Dipole Electric Field

[Fall 2007 Classical (Morning), Problem 5; Griffiths 3.49] The electric potential of a dipole (with moment \( p \)) is

\[
V_{\text{dip}} = \frac{1}{4\pi \varepsilon_0} \frac{p \cdot \hat{r}}{r^2}. \tag{4.6.1}
\]

Calculate the electric field in spherical polar coordinates (with \( p \) aligned along the \( z \)-axis). An electric charge is released from rest at a point on the \( xy \)-plane. Show that it swings back and forth in a semi-circular arc, as though it were a pendulum supported at the origin. The expression for the gradient in spherical coordinates is

\[
\nabla f = \frac{\partial f}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial f}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \hat{\phi}. \tag{4.6.2}
\]

**SOLUTION:** (Thanks to Victoria)

In the case at hand, the potential is

\[
V_{\text{dip}} = \frac{p \cos \theta}{4\pi \varepsilon_0 r^2}. \tag{4.6.3}
\]

Taking the negative gradient gives

\[
E = -\nabla V_{\text{dip}} = \frac{p}{4\pi \varepsilon_0 r^3} (2 \cos \theta \hat{r} + \sin \theta \hat{\theta}). \tag{4.6.4}
\]

The force felt by a charge \( q \) is just \( F = qE \). The plan is to show that the force felt by a pendulum is formally equivalent.

Consider a pendulum of length \( \ell \) and mass \( m \) pivoted at the origin of coordinates with vertical being the \( z \)-direction. \( \theta \) is the angle with respect to the positive \( z \)-axis, but the usual pendulum angle is actually \( \phi = \pi - \theta \), since if the pendulum were to just hang freely, it would hang along the negative \( z \)-axis.

After falling a vertical distance \( \ell \cos \phi \), the pendulum’s speed is \( v^2 = 2g\ell \cos \phi \) (assuming it starts at \( \phi = \pi/2 \), which is on the \( xy \)-plane). Balancing radial forces gives the tension \( T = mg \cos \phi + mu^2/\ell = -3mg \cos \theta \) (after using \( \cos \phi = -\cos \theta \)). Hence the total force is

\[
F = -mg(\cos \theta \hat{r} - \sin \theta \hat{\theta}) + 3mg \cos \theta \hat{r} = mg(2 \cos \theta \hat{r} + \sin \theta \hat{\theta}). \tag{4.6.5}
\]

Thus, the charge moving in the dipole potential at radial distance \( r \) is formally equivalent to a pendulum of length \( r \) and mass \( m = qp/4\pi \varepsilon_0 gr^3 \).

**Aside:** It’s nice to know that there is a relatively simple way to derive the dipole field in case you forget it: just calculate the field produced by two opposite charges, \( \pm q \), slightly displaced by some small distance, \( d \). The dipole moment is \( p = qd \) and points from \(-q\) to \(+q\). In the end, write the field in terms of \( p \).
4.7 Conducting Sphere and Charged Plane

A charged plane lies at \( z = h \). Centered at the origin is a conducting sphere of radius \( R < h \). Consider polar coordinates on the plane with \( \rho \) as the radial distance from the \( z \)-axis. The surface charge density on the plane is

\[
\sigma(\rho) = \frac{R^3}{(h^2 + \rho^2)^{3/2}} \sigma_0, \tag{4.7.1}
\]

where \( \sigma_0 \) is some constant.

Calculate the pressure on the plane as a function of the position on the plane.

**SOLUTION:**

Let us remind ourselves of the image charge setup for a charge \( q \) outside a conducting sphere a distance \( r \) from the sphere’s center. The image charge, \( q' \), must lie on the same line connecting \( q \) and the center of the sphere and must lie on the same side of the center as does \( q \). Let \( q' \) lie a distance \( r' \) from the center. Requiring that the potential vanish on the two points of intersection between the sphere and the line connecting \( q, q' \) and the sphere’s center yields the equations

\[
\frac{q}{r - R} + \frac{q'}{R - r'} = 0, \quad \frac{q}{r + R} + \frac{q'}{r' + R} = 0. \tag{4.7.2}
\]

The solution is

\[
q' = -\frac{R}{r} q, \quad r' = \frac{R^2}{r}. \tag{4.7.3}
\]

Let \((\rho, \phi, h)\) be an arbitrary point on the plane. The radial distance from this point to the sphere’s center and from the image point to the sphere’s center are

\[
r = \sqrt{h^2 + \rho^2}, \quad r' = \frac{R^2}{h^2 + \rho^2}. \tag{4.7.4}
\]

Let \( \theta \) be the polar spherical angle of the point on the plane, which is given by

\[
\tan \theta = \frac{\rho}{h} \quad \implies \quad \cos \theta = \frac{h}{\sqrt{h^2 + \rho^2}}. \tag{4.7.5}
\]

Therefore, we could write

\[
r' = \frac{R^2}{h} \cos \theta. \tag{4.7.6}
\]

Consider the distance between this image point and the point on the \( z \)-axis at \( z = \frac{R^2}{2h} \). By the cosine rule, this distance is

\[
R' = \sqrt{\left(\frac{R^2}{2h}\right)^2 + r'^2 - 2 \frac{R^2}{2h} r' \cos \theta}
= \sqrt{\left(\frac{R^2}{2h}\right)^2 + \left(\frac{R^2}{h}\right)^2 \cos^2 \theta - \frac{R^2}{h} \left(\frac{R^2}{h} \cos \theta\right) \cos \theta}
= \frac{R^2}{2h}, \tag{4.7.7}
\]

since the last two terms cancel each other.

This distance is constant! It follows that the locus of image points, corresponding to all of the points on the plane, forms a sphere of radius \( R' = \frac{R^2}{2h} \) centered at \( z = R' = \frac{R^2}{2h} \). Let \( \theta' \) be the appropriate polar angle and \( \phi' \) the azimuthal angle for this latter sphere with origin taken to be \( z = R' \). Then a differential bit of area on the surface of this image sphere is

\[
dA' = R'^2 \sin \theta' \, d\theta' \, d\phi'. \tag{4.7.8}
\]
Simple geometry gives \( \theta' = 2\theta \) and thus \( \sin \theta' \, d\theta' = 4 \cos \theta \, \sin \theta \, d\theta \). Using the relation between \( \theta \) and \( \rho \) given in Eqn. (4.7.5), we may write this as

\[
\sin \theta' \, d\theta' = \frac{4h^2}{(h^2 + \rho^2)^2} \, \rho \, d\rho.
\]  
(4.7.9)

Obviously, \( \phi' = \phi \). Plug \( R' = \frac{h^2}{2\pi} \) and Eqn. (4.7.9) into Eqn. (4.7.8):

\[
dA' = \frac{R^4}{(h^2 + \rho^2)^2} \, \rho \, d\rho \, d\phi = \frac{R^4}{(h^2 + \rho^2)^2} \, dA,
\]  
(4.7.10)

where \( dA = \rho \, d\rho \, d\phi \) is a bit of area on the plane.

Finally, the plane charge and image charge are related via Eqn. (4.7.3):

\[
dq' = -\frac{R}{r} \, dq = -\frac{R}{\sqrt{h^2 + \rho^2}} \, dq.
\]  
(4.7.11)

Therefore, the surface charge densities are related by

\[
\sigma' = \frac{dq'}{dA'} = -\frac{R}{\sqrt{h^2 + \rho^2}} \, dq \left(\frac{h^2 + \rho^2}{R^2} \right)^2 = -\frac{(h^2 + \rho^2)^{3/2}}{R^3} \, \sigma = -\sigma_0,
\]  
(4.7.12)

where we have plugged in \( \sigma \) given by Eqn. (4.7.1).

Now, we see the point of the seemingly contrived form for \( \sigma \): it ensures that the locus of image charges is a sphere of uniform charge density \(-\sigma_0\).
Chapter 5

Electric Fields in Matter

- Polarizability matrix: $p_i = \alpha_{ij} E_j$.
- Torque on a dipole: $\mathbf{N} = \mathbf{p} \times \mathbf{E}$. Force on a dipole: $\mathbf{F} = (\mathbf{p} \cdot \nabla)\mathbf{E}$.
- The potential and field produced by a polarized object is the same as that produced by a volume charge density $\rho_b = -\nabla \cdot \mathbf{P}$ plus a surface charge density $\sigma_b = \mathbf{P} \cdot \hat{n}$, where $\mathbf{P}$ is the dipole moment density.
- Uniformly polarized sphere: field inside is $\mathbf{E} = -\mathbf{P}/3\varepsilon_0$ and potential outside is the same as a perfect dipole with moment $\mathbf{p} = \frac{4}{3} \pi R^3 \mathbf{P}$.
- Electric displacement $\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$, whose divergence is just the free charges (charges not due to polarization): $\nabla \mathbf{D} = \rho_f$. [Warning: $\nabla \times \mathbf{D} = \nabla \times \mathbf{P}$ since, in electrostatics, $\nabla \times \mathbf{E} = 0$. Thus, unless $\nabla \times \mathbf{P} = 0$, there is generically no potential for $\mathbf{D}$.]
- Electric susceptibility for linear dielectrics: $\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E}$. Then, $\mathbf{D} = \varepsilon \mathbf{E}$, where the permittivity is $\varepsilon = \varepsilon_0 (1 + \chi_e)$. [Warning: even though $\nabla \times \mathbf{E} = 0$, and $\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E}$, it does not necessarily follow that $\nabla \times \mathbf{P} = 0$ since you may have regions with different dielectric constants.]
- Dielectric constant: $\varepsilon = \varepsilon_r = 1 + \chi_e = \varepsilon / \varepsilon_0$.
- Energy density: $\frac{1}{2} \mathbf{D} \cdot \mathbf{E}$.
- Force on dielectric in a capacitor: $F = \frac{1}{2} \frac{dC}{dx}$, where $x$ is how far the dielectric extends into the capacitor.
5.1 Torque on Dipole above a Conducting Plane

[Griffiths 4.6] A perfect dipole $\mathbf{p}$ lies a distance $z$ above an infinite grounded conducting plane. The dipole makes an angle $\theta$ with the perpendicular to the plane. Find the torque on $\mathbf{p}$. If the dipole is free to rotate, in what direction will it come to rest?

\begin{figure}[h]
\centering
\includegraphics[width=0.2\textwidth]{figure5_1}
\caption{Torque on a dipole above a grounded conducting plane.}
\end{figure}

\textbf{SOLUTION:}

This problem should scream “Image charge!”, except we actually need an image dipole. Momentarily imagine the dipole is made up of a displaced positive and negative charge. Then the image dipole, $\mathbf{p}_i$, becomes clear:

\begin{figure}[h]
\centering
\includegraphics[width=0.2\textwidth]{figure5_2}
\caption{Mirror dipole.}
\end{figure}

Let $z$ be the vertical direction and $\hat{y}$ point to the right. Then, the dipole and its image dipole are

\begin{align*}
\mathbf{p} &= p(\sin \theta \hat{y} + \cos \theta \hat{z}), & \mathbf{p}_i &= p(- \sin \theta \hat{y} + \cos \theta \hat{z}).
\end{align*}

The field produced by $\mathbf{p}_i$ at $\mathbf{p}$ is

\begin{equation}
\mathbf{E} = \frac{3(\mathbf{p}_i \cdot \hat{z})\hat{z} - \mathbf{p}_i}{4\pi \varepsilon_0 (2z)^3} = \frac{p(\sin \theta \hat{y} + 2 \cos \theta \hat{z})}{32\pi \varepsilon_0 z^3}.
\end{equation}

Thus, the torque on $\mathbf{p}$ is

\begin{equation}
\mathbf{N} = \mathbf{p} \times \mathbf{E} = \frac{p^2 \sin 2\theta \hat{x}}{64\pi \varepsilon_0 z^3}.
\end{equation}

This points out of the page (counterclockwise rotation) if the prefactor of $\hat{x}$ is positive and into the page (clockwise rotation) if it’s negative. For $0 < \theta < \pi/2$, the dipole rotates counterclockwise and for $\pi/2 < \theta < \pi$, it rotates clockwise. Thus, the stable orientations are pointing directly towards or away from the plane.
5.2 Dielectric Oil

[Griffiths 4.28] Two long coaxial cylindrical metal tubes (inner radius $a$, outer radius $b$) stand vertically in a tank of dielectric oil (susceptibility $\chi_e$, mass density $\rho$). The inner one is maintained at potential $V$, and the outer one is grounded. To what height, $h$, does the oil rise in the space between the tubes?

\begin{figure}[h]
\centering
\includegraphics[width=0.2\textwidth]{dielectric_oil.png}
\caption{Dielectric oil.}
\end{figure}

**SOLUTION:**

**Method 1 (Forces):** Imagine that I have an annular piston that fits right in between the cylinders and I push down on the oil with a force $F_{\text{me}}$. If the level of the oil rises by $dh$ (this is taken to be positive for a rising oil level), then the work I do is actually $dW = F_{\text{me}} dh + V dQ$. If the potential on the cylinders is kept constant, then there are two work terms: mine and one from whatever battery is keeping the potential constant. The work that the battery does is due to the fact that the total charge on the cylinders will change:

$$dW = F_{\text{me}} dh + V dQ. \quad (5.2.1)$$

Now, by Newton’s third law, the electric force attracting the oil into the capacitor is negative of the force I exert: $F_e = -F_{\text{me}}$. Thus,

$$F_e = -\frac{dW}{dh} + V \frac{dQ}{dh} = -\frac{1}{2} \frac{dC}{dh} V^2 + \frac{dC}{dh} V^2 = \frac{1}{2} \frac{dC}{dh} V^2,$$  

where $W = \frac{1}{2} CV^2$ is the familiar energy stored in a capacitor, and we note that $V$ is not differentiated since it remains constant.

So, we need to find the capacitance. Let the cylinder have length $\ell$. Place $+Q$ charge on the inner cylinder, and $-Q$ on the outer one. Gauss’ law implies $D = Q/2\pi\ell r$, pointing radially outward. Thus, the potential is $V(r) = -\frac{Q}{2\pi\ell} \ln r$ and the potential difference between the cylinders is $V = \frac{Q}{2\pi\ell} \ln(b/a)$. Therefore, the capacitance per length is $c = Q/V = 2\pi\epsilon/\ln(b/a) = 2\pi\epsilon_0(1 + \chi_e)/\ln(b/a)$. The capacitance of the region in air is $C_1 = 2\pi\epsilon_0(\ell - h)/\ln(b/a)$ and in oil is $C_2 = 2\pi\epsilon_0(1 + \chi_e)h/\ln(b/a)$. In the setup, these two are in parallel (why?), and so the total capacitance is

$$C = C_1 + C_2 = \frac{2\pi\epsilon_0(\chi_e h + \ell)}{\ln(b/a)}. \quad (5.2.3)$$

Therefore, the upward electric force is

$$F_e = \frac{1}{2} V^2 \frac{dC}{dh} = \frac{\pi\epsilon_0\chi_e V^2}{\ln(b/a)}.$$

$$F_e = \frac{1}{2} V^2 \frac{dC}{dh} = \frac{\pi\epsilon_0\chi_e V^2}{\ln(b/a)}. \quad (5.2.4)$$

This upward force must balance the downward gravitational force $F_g = \pi(b^2 - a^2)\rho gh$. Solving for $h$ yields

$$h = \frac{\epsilon_0\chi_e V^2}{\rho g(b^2 - a^2) \ln(b/a)}. \quad (5.2.5)$$
Aside: There is a good discussion about forces on dielectrics around p. 193 of Griffiths. There, he calculated it for the case when the charge on the capacitor remains the same. The result is identical, but keeping the charge constant is actually simpler: You do not have to take into account an external battery since there isn’t one!

Method 2 (Energy): The energy in the fields is

\[ U_{\text{field}} = \frac{1}{2} \cdot \frac{2\pi \varepsilon_0 (1 + \chi_\varepsilon) h}{\ln(b/a)} \cdot V^2 + \frac{1}{2} \cdot \frac{2\pi \varepsilon_0 (\ell - h)}{\ln(b/a)} \cdot V^2 = \frac{\pi \varepsilon_0 (\chi_\varepsilon h + \ell) V^2}{\ln(b/a)}. \]  

(5.2.6)

The batter must move \( Q = CV \) charge across the potential \( V \), thereby doing work \( W = QV = CV^2 = 2U_{\text{field}} \). Doing so decreases the energy of the battery by this amount. Let \( U_0 \) be the initial energy content of the battery. Then,

\[ U_{\text{battery}} = U_0 - 2U_{\text{field}}. \]  

(5.2.7)

The gravitational potential energy of the system is

\[ U_{\text{gravity}} = \pi (b^2 - a^2) \rho g \int_0^h h' \, dh' = \frac{\pi}{2} (b^2 - a^2) \rho gh^2. \]  

(5.2.8)

The total system energy is the sum of all three: \( U = U_{\text{field}} + U_{\text{battery}} + U_{\text{gravity}} \). Set the derivative of \( U \) with respect to \( h \) equal to zero to find the minimum:

\[ \frac{dU}{dh} = -\frac{\pi \varepsilon_0 \chi_\varepsilon V^2}{\ln(b/a)} + \pi (b^2 - a^2) \rho gh = 0. \]  

(5.2.9)

The solution is

\[ h = \frac{\varepsilon_0 \chi_\varepsilon V^2}{\rho g (b^2 - a^2) \ln(b/a)}. \]  

(5.2.10)
Chapter 6

Magnetostatics

The diagram below (Figure 2.35 in Griffiths) pretty much encapsulates electrostatics.

Figure 6.1: Summary of magnetostatics relations.

- Lorentz force law: $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$. Magnetic forces do no work!
- Magnetic force on volume current: $\mathbf{J} = d\mathbf{I}/da$ (current per unit area-perpendicular-to-flow), then $d\mathbf{F}_{\text{mag}} = (\mathbf{J} \times \mathbf{B}) \, d\tau$.
- Continuity equation: $\nabla \cdot \mathbf{J} = -\partial \rho/\partial t$.
- Coulomb gauge: $\nabla \cdot \mathbf{A} = 0$ turns Ampère’s law into $\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$. As usual, $\mathbf{A}(\mathbf{r}) = \mu_0 \int \frac{\mathbf{J}(\mathbf{r}')}{r} \, d\tau'$ (assuming $J \to 0$ at $\infty$). This comes from the fact that $-\frac{1}{4\pi} \int \frac{1}{r} \, d\tau'$ is Green’s function of the Laplacian, which in turn comes from the identity $-\frac{1}{4\pi} \nabla^2 \frac{1}{r} = \delta(\mathbf{r})$ (the $\nabla$ is w.r.t. $r$).
- $\mathbf{B}_{\text{above}} - \mathbf{B}_{\text{below}} = \mu_0 (\mathbf{K} \times \hat{\mathbf{n}})$.
- Dipole: $\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi r^2} \int \mathbf{r}' \cos \theta' \, d\ell' = \frac{\mu_0 m \times \hat{r}}{4\pi r^2}$, where $m = I \int d\mathbf{a}$ is the magnetic dipole moment. The field can be written $\mathbf{B}_{\text{dip}} = \frac{\mu_0}{4\pi r} [3(\mathbf{m} \cdot \hat{r})\hat{r} - \mathbf{m}] + \frac{2\mu_0}{3} \mathbf{m} \delta(\mathbf{r})$. If you want to derive the delta function, see Griffiths 5.59, p.253-254.
6.1 Current Loop and Solenoid

(Spring 2008 Classical (Afternoon), Problem 4; Griffiths 5.10) Magnetic fields from current loops.

(a) Find the magnitude of the magnetic field at a position \((0, 0, z)\) above the center of a circular loop of radius \(R\) with a steady current \(I\). The loop is in the \(xy\)-plane and centered at the origin.

(b) Find the magnitude of the magnetic field at a point \(P\) on the axis of a tightly wound solenoid consisting of \(n\) turns per unit length with radius \(R\) and carrying current \(I\). Express your answer in terms of \(\theta_1\) and \(\theta_2\), as shown in the figure.

\[\text{Figure 6.2: Spring 2008 Classical (Afternoon), Problem 4}\]

**SOLUTION:** (Thanks to Aaron)

(a) Let the current be \(I = I\hat{\phi}\) so that \(I\, d\ell = IR\, d\phi \hat{\phi}\). The displacement vector that takes us from a source point on the loop to the field point on the axis is \(\mathbf{r} = -R\hat{r} + z\hat{z}\). Thus, Biot-Savart reads

\[
\mathbf{B} = \frac{\mu_0 I}{4\pi} \left\{ \frac{1}{\mathbf{r}} \times \hat{z} \right\} = \frac{\mu_0 I}{4\pi(R^2 + z^2)^{3/2}} \left[ R^2\hat{z} \int_0^{2\pi} d\phi + Rz \int_0^{2\pi} \hat{r} d\phi \right],
\]

where we used the facts \(\hat{\phi} \times \hat{r} = -\hat{z}\) and \(\hat{\phi} \times \hat{z} = \hat{r}\).

Cylindrical symmetry implies that the magnetic field along the axis point directly along the axis. Therefore, we know that the integral involving radial component must vanish. You can just neglect. However, I at least wanted to write (6.1.1) to highlight the fact that the unit vector \(\hat{r}\) depends on \(\phi\) and hence cannot be taken out of the integral, unlike \(\hat{z}\). If you really wanted to, you could write \(\hat{r}\) in Cartesian form and you will find that the integral indeed vanishes. The final result is

\[
\mathbf{B} = \left( R^2 \right) \left( R^2 + z^2 \right)^{-3/2} \hat{z},
\]

(6.1.2)

(b) Chop the cylinder up into a large number of loops of width \(dz\). The current contained in this loop is \(dI = nI\, dz\). Let \(z_1\) be the distance between the point \(P\) and the center of the right end of the solenoid, the \(z_2\) to the left end. Then, adding up the contributions from all the differential loops yields

\[
\mathbf{B} = \frac{\mu_0 n I R^2}{2} \hat{z} \int_{z_1}^{z_2} \frac{dz}{(R^2 + z^2)^{3/2}} = \frac{\mu_0 n I}{2} \hat{z} \int_{z_1/R}^{z_2/R} \frac{du}{(1 + u^2)^{3/2}},
\]

where \(u \equiv z/R\).

Let’s make a trigonometric substitution: \(u = \cot \theta\), which precisely coincides with the definition of the angles in the figure. Then, we have \(du = -\csc^2 \theta \, d\theta\) and \((1 + u^2)^{-3/2} = (1 + \cot^2 \theta)^{-3/2} = \sin^3 \theta\). Hence,

\[
\mathbf{B} = -\frac{\mu_0 n I}{2} \hat{z} \int_{\theta_1}^{\theta_2} \frac{\sin \theta \, d\theta}{\cos \theta} = \frac{\mu_0 n I}{2} \cos \theta \bigg|_{\theta_1}^{\theta_2} = \frac{\mu_0 n I}{2} (\cos \theta_2 - \cos \theta_1) \hat{z},
\]

(6.1.4)
6.2 Current Pinch Effect

[Griffiths 5.38] It may have occurred to you that since parallel currents attract, the current within a single wire should contract into a tiny concentrated stream along the axis. Yet in practice the current typically distributes itself quite uniformly over the wire. How do you account for this? If the positive charges (density $\rho_+$) are at rest, and the negative charges (density $\rho_-$) move at speed $v$ (and none of these depends on the distance from the axis), show that $\rho_-=\gamma^2\rho_+$, where $\gamma=1/\sqrt{1-(v/c)^2}$ and $c^2=1/\mu_0\varepsilon_0$. If the wire as a whole is neutral, where is the compensating charge located? [Notice that for typical velocities, the two charge densities are essentially unchanged by the current (since $\gamma \approx 1$). In plasmas, however, where the positive charges are also free to move, this so-called pinch effect can be very significant.]

**SOLUTION:**

Suppose the negative charges flow through a cylinder that is ever so slightly narrower than the actual wire, due to pinching. We are told to assume that the charge densities are still constant, at least, but now, $\rho_+ < |\rho_-|$ since the negative charges are slightly more bunched up. Therefore, not only do we have a magnetic field inside the wire, we also have an electric field. The magnetic field is azimuthal and the electric field is radial; they both exert a radial force on a mobile negative charge, but in opposite directions. Thus, equilibrium is reached when they cancel.

Using Ampère's law, the magnetic field at radial distance $r$ (within the negative charge density cylinder) is

$$\int B \cdot d\ell = 2\pi r B = \mu_0 I_{\text{enc}} = -\mu_0 \rho_- v \pi r^2 \quad \implies \quad B = -\frac{1}{2} \mu_0 \rho_- vr \hat{\phi}, \quad (6.2.1)$$

where we used the fact that the current density is $J = \rho_- \hat{v}$ with $\rho_-$ negative. [Note: if the current is taken to flow in the $+\hat{z}$ direction, then $v = -v\hat{z}$, since negative charge flow and current are opposite.]

Using Gauss' law, the electric field at radial distance $r$ (within the wire) is

$$\int E \cdot da = 2\pi r LE = \frac{1}{\varepsilon_0} Q_{\text{enc}} = \frac{\rho_+ + \rho_-}{\varepsilon_0} \pi r^2 L \quad \implies \quad E = \frac{\rho_+ + \rho_-}{2\varepsilon_0} r \hat{r}. \quad (6.2.2)$$

For the Lorentz force on a moving charge to vanish, we need $E = -v \times B$:

$$\frac{1}{2\varepsilon_0}(\rho_+ + \rho_-)r \hat{r} = -(-v\hat{z}) \times \left(-\frac{1}{2} \mu_0 \rho_- vr \hat{\phi}\right) = \frac{1}{2} \mu_0 \rho_- v^2 r \hat{r}. \quad (6.2.3)$$

Solving for $\rho_-$ in terms of $\rho_+$ yields $\rho_- = -\gamma^2 \rho_+$, as desired.

The whole wire is still neutral; there is just a very thin outer layer of excess positive charge and a solid cylinder (the whole wire minus the thin outer layer) of excess negative charge.
6.3 Radial Trajectory in Magnetic Field

[Griffiths 5.41] A circularly symmetrical magnetic field (B depends only on the distance from the axis), pointing perpendicular to the page, occupies the shaded region. If the total flux is zero, show that a charged particle that starts out at the center will emerge from the field region on a radial path (provided it escapes at all). On the reverse trajectory, a particle fired at the center from outside will hit its target, though it may follow a weird route getting there.

Figure 6.3: Radial trajectory in a perpendicular magnetic field.

SOLUTION:

We will show that the particle emerges with zero angular momentum (of course, since it starts at the center, it had zero angular momentum to begin with.) The torque is \( \mathbf{N} = q \mathbf{r} \times (\mathbf{v} \times \mathbf{B}) \) and the angular momentum is the time integral

\[
\mathbf{L} = \int \mathbf{N} \, dt = q \int \mathbf{r} \times (d\ell \times \mathbf{B}),
\]

where we have written \( \mathbf{v} = d\ell / dt \).

We can use a vector identity to write this as

\[
\mathbf{L} = q \int (\mathbf{r} \cdot \mathbf{B})d\ell - q \int \mathbf{B} \cdot (\mathbf{r} \times d\ell) = -q \int \mathbf{B} \, d\mathbf{r} = -\frac{q}{2\pi} \int \mathbf{B} \, da,
\]

where the final integral is over the shaded region, and we inserted \( \frac{1}{2\pi} \int_0^{2\pi} d\phi = 1 \), since the integrand is independent of the angular variable, \( \phi \), anyway.

This shows that \( \mathbf{L} \) is directly proportional to the magnetic flux through the region. Since the latter vanishes, so must the former. Hence, the particle leaves the region with no angular momentum and thus, its path must be purely radial.
6.4 Gyromagnetic Ratio

[Griffiths 5.56] A thin uniform donut, carrying charge $Q$ and mass $M$, rotates about its axis (the $z$-axis).

(a) Find the ratio of its magnetic dipole moment to its angular momentum. This is called the gyromagnetic ratio. Note, since masses will enter in, it may be advisable to use $\mu$ instead of $m$ for the dipole moment.

(b) What is the gyromagnetic ratio of a uniform spinning sphere?

(c) According to quantum mechanics, the angular momentum of a spinning electron is $\hbar/2$. What, then, is the electron’s magnetic dipole moment?

SOLUTION:

(a) In a time $dt$, how much charge passes through a fixed point through which the donut passes? Suppose it rotates at angular speed $\omega$, then it will have rotated through an angle $d\phi = \omega dt$. The total charge contained in the arc subtending the angle $d\phi$ is $dQ = \frac{Q}{2\pi} R \omega dt = \frac{Q\omega}{2\pi} dt$. Hence, the current is $I = dQ/dt = Q\omega/2\pi$. Therefore, the magnetic dipole moment for counterclockwise rotation is

$$\mu = IA = \frac{Q\omega}{2\pi} \pi R^2 \hat{z} = \frac{Q\omega R^2}{2} \hat{z}. \quad (6.4.1)$$

The moment of inertia is simply $MR^2$ and so the angular momentum is $L = I\omega = M\omega R^2 \hat{z}$. Thus, we have the gyromagnetic ratio

$$g \equiv \frac{\mu}{L} = \frac{Q}{2M} \quad (6.4.2)$$

(b) Note that $g$ is independent of $R$. Therefore, any figure of revolution around some axis has the same $g$. Namely, a sphere is the figure of revolution of a semicircle. Hence, it also has $g = Q/2M$. The figure also need not be a surface. A solid ball is the figure of revolution of a semi-disk and thus has the same gyromagnetic ratio.

(c) $$\mu = gL = (e/2m_e)(\hbar/2) = e\hbar/4m_e.$$ 

Aside: This semiclassical value is actually off by a factor of almost exactly 2. Dirac’s relativistic electron theory got the 2 right, and Feynman, Schwinger, and Tomonaga later calculated tiny further corrections. The determination of the electron’s magnetic dipole moment remains the finest achievement of quantum electrodynamics, and exhibits perhaps the most stunningly precise agreement between theory and experiment in all of physics. Incidentally, the quantity $e\hbar/2m_e$ is called the Bohr magneton.
Magnetic Fields in Matter

- Torque on magnetic dipole: \( \mathbf{N} = \mathbf{m} \times \mathbf{B} \).
- Force on infinitesimal loop with moment \( \mathbf{m} \): \( \mathbf{F} = \nabla (\mathbf{m} \cdot \mathbf{B}) \).
- Paramagnetism: alignment of moment parallel to magnetic field. Thus, attracted towards strong field. Often due to intrinsic magnetic dipole moment of the electron. Pairs of up and down spin electrons tend to cancel, so this effect is mostly observed in odd-electron atoms.
- Diamagnetism: alignment of moment anti-parallel to magnetic field. Thus, repelled away from strong field. Often due to orbital moment of electron. Weaker effect than paramagnetism, so mostly observed in even-electron atoms when the latter is usually absent.
- Vector potential due to magnetization (magnetic dipole density) \( \mathbf{M} \) is given by \( \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{M}(\mathbf{r}') \times \hat{\mathbf{r}}}{r'} \, d\mathbf{r}' \).
- Also, \( \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}_b(\mathbf{r}')}{\hat{\mathbf{r}}} \, d\mathbf{r}' + \frac{\mu_0}{4\pi} \int_S \frac{\mathbf{K}_b(\mathbf{r}')}{\hat{\mathbf{n}}} \, d\mathbf{a}' \), where \( \mathbf{J}_b = \nabla \times \mathbf{M} \) and \( \mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}} \) are the bound volume and surface current densities.
- Auxiliary field \( \mathbf{H} = \mathbf{B} / \mu_0 - \mathbf{M} \), whose curl is just the free current (currents not due to magnetization): \( \nabla \times \mathbf{H} = \mathbf{J}_f \). [Warning: \( \nabla \cdot \mathbf{H} = -\nabla \cdot \mathbf{M} \), so, unless \( \nabla \cdot \mathbf{M} = 0 \), there is generically no vector potential for \( \mathbf{H} \).]
- Magnetic susceptibility: \( \mathbf{M} = \chi_m \mathbf{H} \). Permeability: \( \mu = \mu_0 (1 + \chi_m) \) so that \( \mathbf{B} = \mu \mathbf{H} \).
7.1 Two Aligned Magnetic Dipoles Attract

[Griffiths 6.3] Find the force of attraction between two magnetic dipoles \( \mathbf{m}_1 \) and \( \mathbf{m}_2 \) a distance \( r \) apart and which are aligned parallel to each other and along the line connecting the two. [Extra: if \( \mathbf{m}_2 \) is misaligned by an angle \( \theta \), what is the torque it feels and if it is free to rotate, but not translate, what will be the equilibrium orientation?]

**SOLUTION:**

Let the \( z \)-axis be the line connecting the two dipoles so that \( \mathbf{m}_i = m_i \hat{z} \) and \( \hat{r} = \hat{z} \). The magnetic field produced by \( \mathbf{m}_1 \) at \( \mathbf{m}_2 \) is

\[
\mathbf{B} = \frac{\mu_0}{4\pi r^3} \left[ (3m_1 \hat{z} \cdot \hat{r}) \hat{z} - m_1 \hat{z} \right] = \frac{\mu_0 m_1 \hat{z}}{2\pi r^4}. \tag{7.1.1}
\]

Therefore, the force exerted on the second dipole is

\[
\mathbf{F} = \nabla (\mathbf{m}_2 \cdot \mathbf{B}) = -\frac{3\mu_0 m_1 m_2 \hat{z}}{2\pi r^4}. \tag{7.1.2}
\]

In the misaligned case, say \( \mathbf{m}_2 = m_2 \sin \theta \hat{x} + m_2 \cos \theta \hat{z} \), the torque is

\[
\mathbf{N} = \mathbf{m}_2 \times \mathbf{B} = -\frac{\mu_0 m_1 m_2 \sin \theta \hat{y}}{2\pi r^3}. \tag{7.1.3}
\]

This will always tend to align the dipole with the magnetic field, and hence with the first dipole (clockwise when \( \theta > 0 \) and counterclockwise when \( \theta < 0 \)).
7.2 Uniformly Magnetized Torus with a Gap

[Spring 2010 Classical (Afternoon), Problem 4; Griffiths 6.10] An iron rod of length $L$ and circular cross section (radius $R$), is given a uniform longitudinal magnetization $M$, and then bent around into a circle with a narrow gap (width $w$). Find the magnetic field at the center of the gap, assuming $w << R << L$.

![Figure 7.1: Uniformly magnetized torus with a gap.](image)

SOLUTION:

Method 1: Set up toroidal coordinates as below, where the circle is a longitudinal cross-section of the torus.

![Figure 7.2: Toroidal coordinates.](image)

The magnetization is $M = M \hat{\phi}$ and so the surface bound current is $K_b = M \times \hat{n} = M \hat{\phi} \times \hat{\rho} = M \hat{\eta}$. This just winds around a longitude. Since $M$ is constant, $J_b = \nabla \times M = 0$. This is like the usual torus with surface current density $M$.

The field along the central circular axis of the torus can be computed using Ampère’s law since symmetry implies that the magnetic field there is constant and just winds around the circle: $B_{\text{center}} = B \hat{\phi}$. The line integral of $B_{\text{center}}$ along this central axis is simply $BL$, where $L$ is the length of the rod. The enclosed current is all the current flowing upward through the inner meridian, which is $K_b L = ML$. Hence, the magnetic field for a complete torus is $B_{\text{complete}} = \mu_0 M \hat{\phi} = \mu_0 M$. Incidentally, we could also have said: $J_f = 0$ and $\nabla \cdot M = 0$ so $B/\mu_0 - M = H = 0$, which gives $B = \mu_0 M$.

Using Biot-Savart, you can compute the magnetic field at the center of a circular loop of radius $R$ and current $I$. The result is $B_{\text{loop}} = \mu_0 I/2R$ and the direction is perpendicular to the loop concordant with the right hand rule (wrap your four fingers in the direction of the current and your thumb naturally points in the direction of the magnetic field.) In this case, we must add to $B_{\text{complete}}$ the magnetic field of a loop whose current, $I = kw = Mw$, flows in the opposite direction to the torus’. Hence, the total magnetic field is

$$B = \mu_0 M \left(1 - \frac{w}{2R}\right)$$

(7.2.1)

Method 2: Recall that $H = \frac{1}{\mu_0} B - M$. Let $H_i$ and $B_i$ be in the iron and $H_g$ and $B_g$ be in the gap. $\int H \cdot d\ell = H_i(L - w) + H_g w = I_{f, \text{enc}} = 0$. Thus, $H_i = -\frac{w}{L-w} H_g = -\frac{w}{L-w} \mu_0 B_g$.

$\nabla \cdot B = 0$ implies $\oint B \cdot d\alpha = 0$. Applied to a wafer-thin pillbox straddling the interface between two regions, this implies $B_\perp$ is continuous across the boundary.

Ampère’s law together with the previous statement implies $B_2 - B_1 = \mu_0 (K \times \hat{n})$, where $\hat{n}$ is the normal pointing from 1 to 2.
Symmetry implies if there is a surface current on the gap boundary, it must be in the $\hat{n}$ direction. Hence, $\mathbf{K} \times \hat{n} \propto \hat{\rho}$, which may make $\mathbf{B}_g$ balloon outwards before returning to the other side of the boundary. However, if $w << R$, we may neglect this effect. Then, $B_g = B_i$. Hence,

$$-\frac{w}{L-w} \mu_0 B_g = H_i = \frac{1}{\mu_0} B_i - M = \frac{1}{\mu_0} B_g - M. \quad \text{(7.2.2)}$$

Therefore, $\mathbf{B}_g = (1 - \frac{w}{L}) \mu_0 M$. Do not be fooled into thinking that this result does not depend on $R$. Neglecting the aforementioned “ballooning” effect requires $w << R$ as an assumption, so the $R$-dependence is implicit. However, the two procedures do not quite agree!
7.3 Current-Carrying Wire with Susceptibility

[Griffiths 6.17] A current $I$ flows down a long straight wire of radius $a$. If the wire is made of linear material (copper, say, or aluminum) with susceptibility $\chi_m$, and the current is distributed uniformly, what is the magnetic field a distance $s$ from the axis? Find all the bound currents. What is the net bound current flowing down the wire?

**Solution:**

Symmetry implies that $\mathbf{H}$ goes around in circles, perpendicular to the cylinder axis, and can only vary with the radius: $\mathbf{H} = H(r) \hat{\phi}$. The free volume current density is $\mathbf{J}_f = I/\pi a^2$, and so the free current enclosed in a circle of radius $s$ is

$$I_{f,\text{enc}} = \begin{cases} I(s^2/a^2), & s < a, \\ I, & s > a. \end{cases} \quad (7.3.1)$$

This is equal to the line integral of $\mathbf{H}$, which is $2\pi s H(s)$. Hence,

$$H(s) = \frac{I}{2\pi s} \begin{cases} s^2/a^2, & s < a, \\ 1, & s > a. \end{cases} \quad (7.3.2)$$

The magnetic field is $\mathbf{B} = \mu \mathbf{H} = \mu H \hat{\phi}$:

$$B(s) = \frac{\mu_0 I}{2\pi s} \begin{cases} (1 + \chi_m) s^2/a^2, & s < a, \\ 1, & s > a. \end{cases} \quad (7.3.3)$$

The bound current density is related to the free current density via

$$\mathbf{J}_b = \nabla \times \mathbf{M} = \nabla \times (\chi_m \mathbf{H}) = \chi_m \mathbf{J}_f. \quad (7.3.4)$$

Since the free current density is $\mathbf{J}_f = I/\pi a^2$, we have

$$\boxed{\mathbf{J}_b = \chi_m I/\pi a^2} \quad (7.3.5)$$

The surface current density is

$$\boxed{\mathbf{K}_b = \mathbf{M} \times \mathbf{n} = \chi_m \mathbf{H} \times \hat{r} = -\chi_m \mathbf{I}/2\pi a} \quad (7.3.6)$$

The net bound current vanishes. I suppose this makes sense, or else we would be getting more or less current out one end than we put in the other end!

$$\boxed{I_b = \pi a^2 \mathbf{J}_b + 2\pi a \mathbf{K}_b = 0} \quad (7.3.7)$$
7.4 Magnetic Ring and Pole Toy

[Griffiths 6.25] A familiar toy consists of donut-shaped permanent magnets (magnetization parallel to the axis), which slide frictionlessly on a vertical rod. Treat the magnets as dipoles, with mass \( m_d \) and dipole moment \( \mathbf{m} \).

(a) If you put two back-to-back magnets on the rod, the upper one will “float” - the magnetic force upward balancing the gravitation force downward. At what height, \( z \), does it float?

(b) If you now add a third magnet (parallel to the bottom one), derive an equation for the ratio of the two heights? (You need not solve explicitly.)

**Figure 7.3: Magnetic ring and pole toy.**

**SOLUTION:**

(a) The field produced by one dipole at the other is

\[
\mathbf{B}_1 = \frac{\mu_0}{4\pi z^3} [3(\mathbf{m}\hat{z}\cdot\hat{z})\hat{z} - \mathbf{m}] = (\frac{\mu_0 m}{2\pi z^3})\hat{z}. 
\]

The second dipole must be opposite, or else the two will attract rather than repel (c.f. Problem 7.1). Thus, the force is

\[
\mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B}) = \nabla\left(-m\hat{z}\cdot\frac{\mu_0 m}{2\pi z^3}\hat{z}\right) = \frac{3\mu_0 m^2}{2\pi z^4} \hat{z}. 
\]

Equating this with \(-m_d g \hat{z}\) and solving for \( z \) yields

\[
z = \left(\frac{3\mu_0 m^2}{2\pi m_d g}\right)^{1/4}. 
\]

(b) Let \( a \) be the distance between the lower and middle magnets and \( b \), between the middle and top magnet. The middle magnet is pushed upward by the lower magnet and downward by the upper magnet:

\[
\frac{3\mu_0 m^2}{2\pi a^4} - \frac{3\mu_0 m^2}{2\pi b^4} - m_d g = 0. 
\]

The top magnet is repelled by the middle and attracted to the lower magnet:

\[
\frac{3\mu_0 m^2}{2\pi b^4} - \frac{3\mu_0 m^2}{2\pi a^4} - m_d g = 0. 
\]

Subtracting the two equations, removing all multiplicative factors and solving for \( \zeta \equiv a/b \) gives

\[
2\zeta^4(\zeta + 1)^4 - (\zeta + 1)^4 - \zeta^4 = 0. 
\]

The numerical solution is \( \zeta \approx 0.85 \).
7.5 Interface between two Linear Magnetic Regions

[Griffiths 6.26] At the interface between one linear magnetic material and another, the magnetic field lines bend. Show that \( \tan \theta_2 / \tan \theta_1 = \mu_2 / \mu_1 \), assuming there is no free current at the boundary.

![Diagram of magnetic field at the interface between two linear magnetic regions.]

**SOLUTION:**

Since \( B_{\text{above}} - B_{\text{below}} = \mu (K \times \hat{n}) \), the perpendicular component of \( B \) (i.e. the one parallel to \( \hat{n} \)) is continuous across the interface. Since \( \nabla \times \mathbf{H} = \mathbf{J}_f \), at the interface we have \( \mathbf{H}_{\text{above}}^\parallel - \mathbf{H}_{\text{below}}^\parallel = K_f \times \hat{n} \). Since we are told that \( K_f = 0 \), it follows that the component of \( \mathbf{H} \) parallel to the interface is also continuous across the interface. Thus,

\[
\tan \theta_2 = \frac{B_2^\parallel}{B_2^\perp} = \frac{\mu_2 H_2^\parallel}{B_1^\perp} = \frac{\mu_2 H_1^\parallel}{\mu_1 B_1^\perp} = \frac{\mu_2}{\mu_1} \tan \theta_1. \tag{7.5.1}
\]
Chapter 8

Electrodynamics

• Ohm’s law: \( J = \sigma f \approx \sigma E \), where \( f \) is the force per unit charge, which is the Lorentz force, and the \( \mathbf{v} \times \mathbf{B} \) can ordinarily be neglected. This implies \( V = IR \), where the resistance, \( R \), is a geometric quantity.

• Drude model: \( \sigma = ne^2\ell/2m_e v_{th} \) where \( n \) is the number density of free electrons, \( \ell \) is the mean free path in the direction of current flow, and \( v_{th} \) is the mean thermodynamic speed of the free electrons.

• Faraday’s law: \( \mathbf{v} \times \mathbf{E} = \mathbf{J} \). Also, \( \mathcal{E} = -d\Phi_B/dt \).

• Mutual inductance: \( i = M_{ij}I_j \) and \( M_{ij} \) is symmetric as is clear from the Neumann formula \( M_{ij} = \mu_0 \int \int \frac{dI_i \cdot dI_j}{r} \). A current in one will induce a current in the other à la Faraday: \( \mathcal{E}_i = -M_{ij}I_j \).

• Self inductance: \( \Phi_i = LI_i \). Again, there is an induced current that opposes changes in \( I \): back emf is \( \mathcal{E} = -L\dot{I} \).

• Energy density in magnetic field: \( \frac{1}{2} \mathbf{A} \cdot \mathbf{J} \) (in currents) or \( \frac{1}{2\mu_0} \mathbf{B}^2 \) (in field).

• Impedances: \( Z_R = R \) (resistor), \( Z_C = -i/\omega C \) (capacitor), \( Z_L = i\omega L \). Impedances add in series and their inverses add in parallel. Resistance is \( R = \text{Re} Z \) and reactance is \( X = \text{Im} Z \).

• Correction to Ampère’s law: \( \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \mathbf{E} \).

• Maxwell’s equations in matter:
  \[
  \nabla \cdot \mathbf{D} = \rho_f, \quad \nabla \times \mathbf{E} = -\dot{\mathbf{B}}, \\
  \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{H} = \mathbf{J}_f + \dot{\mathbf{D}}.
  \]

• Boundary conditions:
  \[
  D_1^i - D_2^i = \sigma_f, \quad E_1^i - E_2^i = 0, \\
  B_1^i - B_2^i = 0, \quad H_1^i - H_2^i = K_f \times \hat{n}.
  \]
 CHAPTER 8. ELECTRODYNAMICS

8.1 Circuit in a Parallel-Plate Capacitor

[Griffiths 7.6] A rectangular loop of wire is situated so that one end (height $h$) is between the plates of a parallel-plate capacitor, oriented parallel to the field $E$. The other end is way outside, where the field is essentially zero. What is the emf in this loop? If the total resistance is $R$, what current flows?

![Figure 8.1: Circuit in a parallel-plate capacitor.](image)

**SOLUTION:**

This is somewhat of a trick question. Suppose that the potential just below the resistor is $V$ and that just above is 0. Then, the line integral of the electric field from the bottom, around the loop (clockwise) and to the top is given by $V = -\int E \cdot d\ell$. Since the resistor is in a region where the electric field vanishes, we can add the line integral of $E$ across the resistor with impunity, since it vanishes anyway. This would give us that the emf is $V = -\int E \cdot d\ell$. But, Stoke's theorem would imply $V = -\int (\nabla \times E) \cdot da = 0$, since all electrostatic fields have vanishing curl! It follows that there is no emf and thus no current.
8.2 Morphing Rectangular Loop Around a Wire

[Kevin G.] A rectangular loop of width $a_0$, length $\ell$ and resistance $R$ sits a distance $b_0$ away from an infinite straight wire carrying a constant current $I$ (both lie on the same plane.) The following things happen simultaneously for $t \geq 0$: (1) the loop is moved directly away from the wire at the constant rate $\dot{b}$; and (2) the loop’s width is increased at the constant rate $\dot{a}$ and the length $\ell$ is unchanged. If there is to be no induced current around the loop, solve for $a(t)$ and $b(t)$.

![Figure 8.2: Morphing rectangular loop around a wire.](image)

**SOLUTION:**

Of course, $a(t) = a_0 + \dot{a}t$ and $b(t) = b_0 + \dot{b}t$ since $\dot{a}$ and $\dot{b}$ are constant. But the requirements of the question will relate $\dot{a}$ and $\dot{b}$ together.

The magnetic field produced by the straight wire is $B(r) = \mu_0 I/2\pi r$ and points into the page in the region of the loop. Let us traverse the loop in the clockwise direction so that $da$ points into the page as well. Call the vertical direction $z$. Then the magnetic flux through the loop as a function of time is

$$
\Phi_B(t) = \frac{\mu_0 I \ell}{2\pi} \int_{b}^{a+b} \frac{dr}{r} = \frac{\mu_0 I \ell}{2\pi} \ln \left(1 + \frac{a(t)}{b(t)}\right).
$$

(8.2.1)

For there to be no induced current, $\Phi_B$ must not vary with time. It follows that $a(t)/b(t)$ must be constant and equal to $a_0/b_0$. This implies

$$\dot{b} = \frac{b_0}{a_0} \dot{a},
$$

(8.2.2)

and thus,

$$a(t) = a_0 \left(1 + \frac{\dot{a}}{a_0} t\right), \quad b(t) = b_0 \left(1 + \frac{\dot{a}}{a_0} t\right).
$$

(8.2.3)
CHAPTER 8. ELECTRODYNAMICS

8.3 Magnet Falling Through Tube

[Kevin G.] Calculate the terminal velocity for an ideal magnetic dipole, with dipole moment \( \mu \) and mass \( m \), falling down the center of an infinitely long conducting (but non-ferromagnetic) cylindrical tube of radius \( R \), thickness \( t \ll R \) and conductivity \( \sigma \). Assume that \( \mu \) aligns with the central axis at all times. [You will need the integral \( \int_{-\infty}^{\infty} x^2(1 + x^2)^{-5/2} \, dx = 5\pi/128 \].]

**SOLUTION:**

Since this is a mathematically taxing problem, let’s first discuss the essential physics. Set up coordinates so that \( \hat{z} \) points vertically upwards, the central axis of the tube is the \( z \)-axis, and set \( \mu = \mu \hat{z} \). The magnetic field produced by the dipole is strong and pointing away from the top and strong and pointing towards the bottom. Hence, as the dipole falls towards a given circular cross-section of the tube below the dipole, the magnetic flux upward through this disk is increasing. It follows that a current is induced in the tube below the dipole is clockwise as in the picture. Similarly, the current induced above the dipole is counterclockwise. Each can be treated like a dipole with moment equal to the area of the disk multiplied by the induced current. The bottom currents are anti-aligned and the top ones are aligned with \( \mu \). Thus, \( \mu \) is repelled by the induced dipoles below it and attracted to the induced dipoles above it. When this force balances gravity, \( mg \), the dipole has reached terminal velocity.

**Method 1:** Recall that the vector potential for a dipole is

\[
A(r') = \frac{\mu_0}{4\pi} \frac{\mu \times \hat{r}'}{\hat{r}''}, \tag{8.3.1}
\]

where \( z = r' - r_0 \), \( r' \) is an arbitrary point and \( r_0 \) is the position of the dipole.

Evaluated on the tube itself,

\[
\hat{r} = \frac{r' - r_0}{|r' - r_0|} = \frac{R\hat{\rho} + (z' - z_0)\hat{z}}{\sqrt{R^2 + (z' - z_0)^2}} \tag{8.3.2}
\]

Thus, the vector potential on the tube is

\[
A(r') = \frac{\mu_0 \mu R}{4\pi} \frac{\hat{\Phi}}{(R^2 + (z' - z_0)^2)^{3/2}}. \tag{8.3.3}
\]

Since there are no electrostatic charges, we can set the scalar potential to zero. Then, the induced electric field on the tube is

\[
E_{\text{ind}} = -\hat{\nabla} = -(v \cdot \nabla_0)A, \tag{8.3.4}
\]

where \( v = \dot{z}_0 \hat{z} = -v\hat{z} \), where \( v \) is the speed of the dipole, and where \( \nabla_0 \) is with respect to the dipole’s coordinates.
8.3. MAGNET FALLING THROUGH TUBE

Evaluating the electric field yields
\[ E_{\text{ind}} = \frac{3\mu_0\mu R v}{4\pi} \frac{z' - z_0}{(R^2 + (z' - z_0)^2)^{5/2}} \hat{\phi}. \] (8.3.5)

This will induce a current in the ±\hat{\phi} direction (as mentioned earlier). To find this current, we
require the emf, which is the line integral of \( E \) around the loop (the circular cross-section of the
tube):
\[ \oint E_{\text{ind}} \cdot d\ell = 2\pi RE_{\text{ind}} = \frac{3\mu_0\mu R^2 v}{2} \frac{z' - z_0}{(R^2 + (z' - z_0)^2)^{5/2}} \hat{\phi}. \] (8.3.6)

Now, to find the current, in a loop of conductor of thickness \( t \ll R \), height \( dz' \) and length \( 2\pi R \),
we must multiply (8.3.6) by the conductance, which is \( G = \sigma t d\xi / 2\pi R \):
\[ dI = \frac{3\mu_0\mu \sigma t R v}{4\pi} \frac{z' - z_0}{(R^2 + (z' - z_0)^2)^{5/2}} d\xi \hat{\phi}. \] (8.3.7)

Using Biot-Savart, one can compute the magnetic field along the axis of a circular current loop. It is
\[ dB = \frac{\mu_0 dI}{2} \frac{R^2 \hat{z}}{(R^2 + (z' - z)^2)^{3/2}}, \] (8.3.8)

where \( z \) is some arbitrary point on the \( z \)-axis.

The differential force exerted on the falling dipole is
\[ d\mathbf{F}_{\text{ind}} = \nabla (\mathbf{\mu} \cdot d\mathbf{B}_{\text{ind}}) \bigg|_{z=z_0}. \] (8.3.9)

After some algebra, this gives
\[ d\mathbf{F}_{\text{ind}} = \frac{9\mu_0^2 \mu^2 \sigma t R^3 v}{8\pi} \frac{(z' - z_0)^2}{(R^2 + (z' - z_0)^2)^5} d\xi \hat{z}. \] (8.3.10)

Integrating over \( z' \) from \(-\infty\) to \(+\infty\), and using the integral given in the problem yields
\[ \mathbf{F}_{\text{ind}} = \frac{45\mu_0^2 \mu^2 \sigma t v}{1024 R^4} \hat{z}. \] (8.3.11)

Setting this equal to \( mg\hat{z} \) and solving for \( v \) yields
\[ v_{\text{term}} = \frac{1024 mg R^4}{45\mu_0^2 \mu^2 \sigma t}. \] (8.3.12)

Aside: In the discussion, we treated each current loop like a dipole, but that was only to
derive the fact that the induced force on the falling dipole is upward. You can’t actually do
the calculation that way since the strongest currents are induced near the dipole at which
the current loops are sufficiently large as to invalidate the approximation.

Method 2: The reason why I wanted to deal directly with the vector potential was so that
I wouldn’t have to actually compute any flux (area integral of the magnetic field of the falling
dipole). It is quite a chore to do this directly. However, we can use a trick: think of the falling
dipole as a magnetic monopole of charge \(-\alpha\) at \( z_0 \) and another one with \(+\alpha\) at \( z_0 + \delta \). The
dipole moment is then, \( \mu = \alpha \delta \). The field of a monopole is the same as that of a point charge
with the changes \( \epsilon_0 \rightarrow 1/\mu_0 \) and \( q \rightarrow \alpha \). Now, it is straightforward to compute the flux due to
each monopole. Add them up and you get
\[ \Phi_B = \frac{\mu_0 \mu}{2} \frac{R^2}{(R^2 + (z' - z_0)^2)^{3/2}}, \] (8.3.13)
Taking the time derivative of this will give us the same emf (8.3.6), from which everything else follows as in method 1.

Aside: It’s quite possible that I botched some minus signs here and there. But, I would never rely on my getting the signs correct just on the basis of the mathematics for an induction problem, anyway. Always let Lenz’ law to take care of the signs for you and, if you want, you can put absolute value signs henceforth. On the flip side, if you find that you get an answer that obviously just has the wrong sign (e.g. if we found the induced force to point downward), then just say at the end what the sign really should be on the basis of Lenz’ law and cavalierly switch it.
8.4 Mutual Inductance

[Kevin G.] A rectangular (or elliptical) loop of wire has width 2a and length ℓ ≫ a. At the center of this loop, and coplanar with it, there is a square loop of wire of side-length a. Suppose the small loop carries a current I. Compute the resulting magnetic flux through the large loop. What is the mutual inductance between the two loops?

\[ \text{(Figure 8.4: Mutual inductance of a pair of loops.)} \]

**SOLUTION:**

It would be quite difficult to directly calculate the flux through the big loop due to the small one. However, by the symmetry of mutual inductance, this flux would be the same as the flux through the small loop if the same current were to flow in the big loop. Since the big loop is very long compared to its width, and the geometry is such that the two long sides are parallel, we can treat the field in the vicinity of the small loop as that produced by two infinite parallel wires with oppositely directed currents, I. Let \( x \) be the distance in the plane above the lower long edge, so that the distance to the upper long edge is \( 2a - x \). Then, the field points out of the page with magnitude

\[
B = \frac{\mu_0 I}{2\pi} \left( \frac{1}{x} + \frac{1}{2a - x} \right).
\]

Then, the flux through the small square is

\[
\Phi = \frac{\mu_0 I a}{2\pi} \int_{a/2}^{3a/2} \left( \frac{1}{x} + \frac{1}{2a - x} \right) dx = \frac{\mu_0 a \ln 3}{\pi} I.
\]

(8.4.2)

As mentioned, this would also have been the flux through the bigger loop had the same current run in the smaller loop.

The mutual inductance is

\[
M = \frac{\mu_0 a \ln 3}{\pi}.
\]

(8.4.3)
8.5 Coaxial Cable

[Fall 2008 Classical (Morning), Problem 3] Consider a coaxial cable with solid inner conductor of radius \( a \), and a thin outer shell conductor of radius \( b \) (the conductors are highly conducting). The region between the conductors is filled with a dielectric of permittivity \( \epsilon \). A current \( I \) flows on the inner conductor and returns on the outer conductor.

(a) What is the inductance per length of the coaxial cable?

(b) What is the capacitance per length of the coaxial cable?

(c) What is the characteristic impedance of the cable, and the speed of wave propagation through it?

\[ L = \frac{\mu_0 \ln(b/a)}{2\pi}, \]

\[ C = \frac{2\pi \epsilon}{\ln(b/a)}. \]

Figure 8.5: Fall 2008 Classical (Morning), Problem 3

SOLUTION:

(a) We need to compute the flux in the dashed rectangular region in the diagram.

The magnetic field is \( B = \frac{\mu_0 I}{2\pi r} \) and so the flux is

\[ \Phi_B = \frac{\mu_0 I \ell}{2\pi} \int_a^b \frac{dr}{r} = \frac{\mu_0 I \ell}{2\pi} \ln(b/a), \]

where \( \ell \) is the length of the cable. The inductance per length is

\[ L = \frac{\mu_0 \ln(b/a)}{2\pi}. \]

(b) Place +\( \lambda \) charge per length on the inner cylinder and -\( \lambda \) on the outer. The electric field in the dielectric is \( E = \frac{\lambda}{2\pi \epsilon r} \) and points radially outward. The line integral of \( \mathbf{E} \) from the inner to the outer cylinder is \( \lambda \ln(b/a)/2\pi \epsilon \), which is the potential difference between the cylinders. Thus, the capacitance per length is

\[ C = \frac{2\pi \epsilon}{\ln(b/a)}. \]
8.5. COAXIAL CABLE

(c) Obviously, the impedance is a function of the frequency of the driving voltage. But, it is reasonable that the dependence should die in some regime (here, it’s for high frequencies.) In that case, we can solve this using dimensional analysis:

\[ Z_0 = \sqrt{L/C}. \] (8.5.4)

where we could use inductance and capacitance or the same quantities per length, since the ratio would be unchanged. This is the only combination with the correct units of impedance. But, let us actually solve the problem.

Method 1 (Fourier): The cable is equivalent to the circuit

![Equivalent circuit for the coaxial cable.]

Start the circuit at some position \( \ell \) (that’s where the source is). Let \( I(\ell) \) be the current entering the first inductor, \( I(\ell + d\ell) \) the second inductor, and so on. Then, the current entering the first capacitor is \( I(\ell) \) and at the first node is \( V(\ell + d\ell) \). Thus, the voltage dropped across the first inductor is \( V(\ell) \) and that dropped across the first capacitor is \( V(\ell + d\ell) \). Thus,

\[ i\omega dL I(\ell) = Z_{dL} dL = V_{dL} = V(\ell) - V(\ell + d\ell) = -dV, \]
\[ \frac{V(\ell + d\ell)}{1/i\omega dC} = \frac{V_{dC}}{Z_{dC}} = I_{dC} = I(\ell) - I(\ell + d\ell) = -dI, \]

Writing \( dL = (L/\ell) d\ell \) and \( dC = (C/\ell)d\ell \), gives

\[ V' = -i\omega(L/\ell)I, \quad I' = -i\omega(C/\ell)V, \]

where the primes indicate a derivative with respect to \( \ell \).

Together, these yield

\[ V'' + \omega^2(L/\ell)(C/\ell)V = 0 \quad \Rightarrow \quad [-k^2 + \omega^2(L/\ell)(C/\ell)]V = 0, \]

(8.5.7)

where \( \tilde{V} \) is the Fourier transform of \( V \).

Hence, the dispersion relation and wave propagation speed are

\[ \omega = \frac{k}{\sqrt{(L/\ell)(C/\ell)}} = \frac{k}{\sqrt{\mu_0 \varepsilon}} \quad \Rightarrow \quad v = \frac{1}{\sqrt{\mu_0 \varepsilon}}, \]

(8.5.8)

which is precisely the speed of light in the dielectric!

The Fourier transform of (8.5.6) reads

\[ -ik\tilde{V} = -i\omega(L/\ell)\tilde{I} \quad \Rightarrow \quad \tilde{V} = \frac{\omega(L/\ell)}{k}\tilde{I}. \]

(8.5.9)
The factor multiplying $\mathbf{I}$ is the characteristic impedance, $Z_0$. Using the dispersion relation for $\omega/k$, we get

$$Z_0 = \sqrt{\frac{L/\ell}{C/\ell}} = \sqrt{\frac{L}{C}} = \left(\frac{\ln(b/a)}{2\pi}\right)\sqrt{\frac{\mu_0}{\epsilon}} \quad (8.5.10)$$

The factor $\sqrt{\mu_0/\epsilon}$ would be the impedance of all space filled with the dielectric. The prefactor is purely geometric.

**Method 2 (Differential Relations):** Consider the diagram

```
      dI                  I+di
       \|
       |   \text{dC}
       |     \text{dV}/dx
       |     \text{dV}/dx
       |     V
       |     V+\text{dV}
```

Figure 8.8: Voltages and currents for the equivalent circuit for the coaxial cable.

Let $V(x, t)$ and $I(x, t)$ be the voltage and current as a function of time and position along the cable. The inductor and capacitor equations respectively yield

$$dV = -\frac{L}{\ell} \frac{dx}{dt} \frac{\partial I}{\partial t} \quad dI = -\frac{C}{\ell} \frac{dx}{dt} \frac{\partial V}{\partial t}. \quad (8.5.11)$$

Note that the negative sign in the inductor equation is standard. The negative sign in the capacitor equation arises here because the current, $dI$, through the capacitor in the diagram is going opposite to the direction that it would go were $V$ to be positive.

The impedance is

$$Z = \frac{\partial V}{\partial I} = \left(-\frac{L}{\ell} \frac{dx}{dt} \frac{\partial I}{\partial t}\right)\left(-\frac{C}{\ell} \frac{dx}{dt} \frac{\partial V}{\partial t}\right)^{-1} = \frac{L/\ell}{C/\ell} \frac{\partial I}{\partial V} = \frac{L/\ell}{C/\ell} Z. \quad \text{Therefore,}$$

$$Z = \sqrt{\frac{L/\ell}{C/\ell}} = \sqrt{\frac{\ln(b/a)}{2\pi}} \sqrt{\frac{\mu_0}{\epsilon}} \quad (8.5.12)$$

To get the propagation speed, we will derive the wave equation:

$$\frac{\partial^2 V}{\partial x^2} = -\frac{L}{\ell} \frac{\partial^2 I}{\partial x \partial t} = -\frac{L}{\ell} \left(-\frac{C}{\ell} \frac{\partial^2 V}{\partial t^2}\right) \quad (8.5.13)$$

Hence, the propagation speed is

$$v = \sqrt{\frac{1}{(L/\ell)(C/\ell)}} = \sqrt{\frac{1}{\mu_0 \epsilon}} \quad (8.5.14)$$

**Aside 1:** Note that we made the change $\epsilon_0 \to \epsilon$. Incidentally, we didn’t change $\mu_0$ because we weren’t told what to change it to and it tends to not vary very much across ordinary materials anyway.

**Aside 2:** We could have computed the impedance of the semi-infinite circuit in Figure 8.7 by replacing $dL$ with $dZ_L = i\omega dL$ and $dC$ with $dZ_C = -i/\omega dC$. Then, due to semi-infinity, if we replace the ladder with its equivalent impedance, $Z_{eq}$, add another $dZ_C$ and $dZ_L$ on the left, and then compute the equivalent impedance of that, $Z'_{eq}$, then $Z'_{eq} = Z_{eq}$. This will give a simple quadratic equation for $Z_{eq}$. 
Chapter 9

Conservation Laws

- Poynting vector: \( \mathbf{S} = \frac{1}{\mu_0} (\mathbf{E} \times \mathbf{B}) \). The rate of work done on the charges in a volume \( \mathcal{V} \) is
  \[ \frac{dW}{dt} = -\frac{dU_{em}}{dt} - \oint_{\partial\mathcal{V}} \mathbf{S} \cdot d\mathbf{a}, \]
  where \( \partial\mathcal{V} \) is the boundary of the region, and \( U_{em} \) is the usual energy stored in the electric and magnetic fields in the region.

- Maxwell stress tensor: \( T_{ij} = \varepsilon_0 (E_i E_j - \frac{1}{2} \delta_{ij} E^2) + \frac{1}{\mu_0} (B_i B_j - \frac{1}{2} \delta_{ij} B^2) \).

- Force on charges in volume \( \mathcal{V} \) due to electromagnetic field is
  \[ F_i = \oint_{\partial\mathcal{V}} T_{ij} d\mathbf{a}_j - \varepsilon_0 \mu_0 \frac{d}{dt} \int_{\mathcal{V}} S_i d\tau. \]

- Momentum density in fields: \( \mu_0 \epsilon_0 \mathbf{S} \).

- Angular momentum density in fields: \( \mu_0 \epsilon_0 \mathbf{r} \times \mathbf{S} \).
9.1 Spinning Charged Spherical Shell

[Griffiths 8.3] Calculate the force of magnetic attraction between the northern and southern hemispheres of a uniformly charged spinning spherical shell, with radius $R$, angular velocity $\omega$, and surface charge density $\sigma$.

**SOLUTION:**

**Method 1:** In the case of electrostatics, the pressure on a surface of charge is $\sigma E_{\text{avg}}$, where $E_{\text{avg}}$ is the average of the electric field just above and below the surface, so, for magnetostatics, the pressure on a surface with current density is $K \times B_{\text{avg}}$. In our case, the surface current density is $K = \sigma v$, where $v$ is the tangential velocity of a point on the sphere.

The vector potential is $A(r) = \frac{\mu_0}{4\pi} \int \frac{K(r') \, da'}{r}$, where $K = \sigma v$, $v$ is the tangential velocity of a point on the sphere.

We can make a $u = \cos \theta'$ substitution and one integration by parts to get

$$A(r) = -\frac{\mu_0 R^3 \sigma \omega \sin \psi}{2} \frac{\cos \theta' \sin \theta' \, d\theta'}{\sqrt{r^2 + r^2 - 2Rr \cos \theta'}} \hat{y}. \quad (9.1.2)$$

The average of the two is

$$B_{\text{avg}} = \frac{\mu_0 R^3 \sigma \omega}{3} (2 \cos \theta \hat{r} + \sin \theta \hat{\theta}). \quad (9.1.3)$$

For the outside:

$$B_{\text{out}} = \frac{\mu_0 R^4 \sigma \omega}{3} \left( \frac{\hat{r}}{r} \frac{\partial}{\partial \theta} \left( \frac{\sin^2 \theta}{r^2} \right) - \frac{\hat{\theta}}{r} \frac{\partial}{\partial r} \left( \frac{\sin \theta}{r} \right) \right) \hat{r} \sin \theta \hat{\theta}. \quad (9.1.4)$$

Alternatively, one could have just remembered that the external field of a rotating charged sphere is the same as the dipole field of the same sphere but now with magnetization $\mathbf{M} = \sigma R \omega \hat{z}$.

The field just outside the surface is

$$B_{\text{out}} = \frac{\mu_0 R \sigma \omega}{3} (2 \cos \theta \hat{r} + \sin \theta \hat{\theta}). \quad (9.1.5)$$

The average of the two is

$$B_{\text{avg}} = \frac{\mu_0 R \sigma \omega}{6} (2 \cos \theta \hat{r} - \sin \theta \hat{\theta}). \quad (9.1.6)$$
Recalling that $K = R \sigma \omega \sin \theta \phi$, we find
\[ K \times B_{\text{avg}} = \frac{1}{6} \mu_0 (R \sigma \omega)^2 \sin \theta (\sin \theta \hat{r} + 4 \cos \theta \hat{\theta}). \tag{9.1.9} \]

The $z$-component is found by writing $\hat{r} = \cdots + \cos \theta \hat{z}$ and $\hat{\theta} = \cdots - \sin \theta \hat{z}$:
\[ (K \times B_{\text{avg}})_z = -\frac{1}{2} \mu_0 (R \sigma \omega)^2 \sin^2 \theta \cos \theta. \tag{9.1.10} \]

This is the only component of the force we need since symmetry implies that the force is in the $z$-direction. The integral of this over the northern hemisphere gives
\[ F = 2 \pi \int_0^{\pi/2} (K \times B_{\text{avg}})_z d\theta = -\frac{1}{4} \pi \mu_0 (R^2 \sigma)^2 \hat{z}. \tag{9.1.11} \]

This force is attractive, unlike the electric field case for a uniformly charged sphere.

**Method 2:** Since the fields are static, the time derivative of the Poynting vector term in the force vanishes. Since the force is only in the $z$-direction, we only need the $z$-components, $T_{zi}da_i$:
\[ T_{zi}da_i = \frac{1}{\mu_0} (B_z B_x da_x + B_z B_y da_y + B_z B_z da_z - \frac{1}{2} B^2 da_z) \]
\[ = \frac{1}{\mu_0} (B_z (B \cdot da) - \frac{1}{2} B^2 da_z). \tag{9.1.12} \]

We have neglected the electric parts since those yield electric forces. We require a surface that encloses the entire northern hemisphere. One is drawn below in red and consists of a red hemisphere glued to a red disk at the equator. This is important: we need to add the forces on both the red hemisphere and the red disk.

![Figure 9.1: Surface for calculation of force on northern hemisphere.](image)

The field on the hemisphere is $B_{\text{out}}$ whose $z$-component and square magnitude are
\[ B_z = \frac{1}{2} \mu_0 R \sigma \omega (3 \cos^2 \theta - 1), \quad B^2 = \frac{1}{4} (\mu_0 R \sigma \omega)^2 (3 \cos^2 \theta + 1). \tag{9.1.13} \]

The differential area vector and its $z$-component are
\[ da = R^2 \sin \theta \, d\theta \, d\phi \, \hat{r}, \quad da_z = R^2 \sin \theta \cos \theta \, d\theta \, d\phi. \tag{9.1.14} \]

The dot product of the magnetic field and the differential area vector is
\[ B \cdot da = \frac{2}{3} \mu_0 R^3 \sigma \omega \sin \theta \cos \theta \, d\theta \, d\phi. \tag{9.1.15} \]

Then, substituting these into (9.1.12) we have
\[ T_{zi}da_i = \frac{1}{15} \mu_0 (R^2 \sigma \omega)^2 (9 \cos^2 \theta - 5) \sin \theta \cos \theta \, d\theta \, d\phi. \tag{9.1.16} \]

Integrating this over the northern hemisphere gives
\[ F_{\text{hemi}} = -\frac{1}{30} \pi \mu_0 (R^2 \sigma \omega)^2 \hat{z}. \tag{9.1.17} \]

The magnetic field on the disk is $B_{\text{in}}$. The same tedious calculation yields
\[ F_{\text{disk}} = -\frac{2}{5} \pi \mu_0 (R^2 \sigma \omega)^2 \hat{z}. \tag{9.1.18} \]

Adding the two yields the total force, which agrees with (9.1.11).
9.2 Rotation due to Demagnetization

(Spring 2009 Classical (Afternoon), Problem 2) Imagine an iron sphere of radius $R$ that carries charge $Q$ (spread uniformly over and glued down to the surface) and a uniform magnetization $\mathbf{M} = M\hat{z}$. The sphere is initially at rest.

(a) Find the electric and magnetic fields.

(b) Suppose the sphere is gradually and uniformly demagnetized by heating it above the Curie point. Find the induced electric field and the torque this exerts on the sphere, and compute the total angular momentum imparted to the sphere in the course of the demagnetization.

(c) Where does this angular momentum come from?

SOLUTION:

(a) Looking at the professor’s solution, I think he or she simply expected you to know the magnetic field of a uniformly magnetized sphere! Firstly, you should definitely know the magnetic field of a perfect magnetic dipole. In the present case, it’s a good thing to remember that the magnetic field outside a uniformly magnetized sphere is the same as that of a perfect dipole whose magnetic moment is equal to the total moment of the sphere (i.e. $\mathbf{m} = \frac{1}{3}\pi R^3 \mathbf{M}$). I guess you would just have to remember $\mathbf{H} = \frac{1}{\mu_0} \mathbf{M}$ inside the sphere and, therefore, $\mathbf{B} = \frac{2}{3}\mu_0\mathbf{M}$. By the way, be warned that the professor mixes up $\mathbf{B}$ and $\mathbf{H}$, I believe. The electric field is easy: it’s just the field of a point charge $Q$ outside and it vanishes inside.

Computing $\mathbf{B}$ “from scratch”:

The magnetic field, auxiliary field and magnetization are related via

$$\mathbf{H} = \frac{1}{\mu_0} \mathbf{B} - \mathbf{M}. \quad (9.2.1)$$

We assume that no net free currents (surface or volume) exist in the iron sphere. Then, $\mathbf{B}$ is divergenceless and $\mathbf{H}$ has zero curl (since $\nabla \times \mathbf{H} = \mathbf{J}_f$). It must therefore be possible to write $\mathbf{H} = -\nabla \Phi_M$ for some magnetic scalar potential, $\Phi_M$. We assume that the permeability of the iron is essentially the same as that of vacuum ($\mu$ tends to vary very little among mundane materials). Thus,

$$0 = \frac{1}{\mu_0} \nabla \cdot \mathbf{B} = \nabla \cdot (\mathbf{H} + \mathbf{M}) = -\nabla^2 \Phi_M + \nabla \cdot \mathbf{M}. \quad (9.2.2)$$

Using the appropriate Green’s function for the Laplacian, we can solve for $\Phi_M$:

$$\Phi_M(\mathbf{r}) = -\frac{1}{4\pi} \int \frac{\nabla' \cdot \mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3r'. \quad (9.2.3)$$

Of course, this comes from the fact that $-\frac{1}{4\pi} \nabla^2 \frac{1}{|\mathbf{r} - \mathbf{r}'|} = \delta(\mathbf{r} - \mathbf{r}')$.

This integral is over all of space, except, of course, $\mathbf{M}$ vanishes outside the sphere. We conclude that we should be able to perform integration by parts to transfer the $\nabla'$ on to the term $|\mathbf{r} - \mathbf{r}'|^{-1}$ without worrying about a boundary term at infinity. In addition, we use the simple relation $\nabla' |\mathbf{r} - \mathbf{r}'|^{-1} = -\nabla |\mathbf{r} - \mathbf{r}'|^{-1}$ to write (9.2.3) as

$$\Phi_M(\mathbf{r}) = -\frac{1}{4\pi} \nabla \cdot \int \frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3r' = -\frac{M}{4\pi} \frac{\partial}{\partial z} \int_0^R r'^2 dr' \int d\Omega' \frac{1}{|\mathbf{r} - \mathbf{r}'|}. \quad (9.2.4)$$
9.2. ROTATION DUE TO DEMAGNETIZATION

We expand $|r - r'|^{-1}$ in products of spherical harmonics:

$$
\frac{1}{|r - r'|} = 4\pi \sum_{\ell,m} \frac{1}{2\ell + 1} \frac{r_<}{r_< + 1} Y_{\ell m}^*(\theta', \phi') Y_{\ell m}(\theta, \phi),
$$

(9.2.5)

where $r_\text{<} = \min(r, r')$ and $r_\text{>} = \max(r, r')$.

With $Y_{00}(\theta', \phi') = 1/\sqrt{4\pi}$, we may write (9.2.4) as

$$
\Phi_M(r) = -M_0 \sqrt{4\pi} \frac{\partial}{\partial z} \sum_{\ell,m} Y_{\ell m}(\theta, \phi) \int_0^r \frac{r_<}{r_< + 1} r^2 \, dr' \, dY Y_{\ell m}^*(\theta', \phi') Y_{00}(\theta', \phi').
$$

(9.2.6)

Finally, we need the orthonormality conditions

$$
\int dY Y_{\ell m}^*(\theta', \phi') Y_{\ell m}(\theta', \phi') = \delta_{\ell r} \delta_{m r'},
$$

(9.2.7)

to write 9.2.6 as

$$
\Phi_M(r) = -M_0 \frac{\partial}{\partial z} \int_0^r \frac{r^2}{r_> \, dr'} = -M_0 \cos \theta \frac{\partial}{\partial r} \int_0^r \frac{r^2}{r_> \, dr'},
$$

(9.2.8)

where use was made of $\partial r/\partial z = \cos \theta$.

In the region outside the sphere, $r > R$, we get

$$
\Phi_M(r) = -\frac{1}{3} M_0 R^3 \cos \theta \frac{\partial}{\partial r} \frac{1}{r} = \frac{1}{3} M_0 R^3 \cos \theta \frac{1}{r^2},
$$

(9.2.9)

which is the potential for a dipole moment $m = (4\pi R^3/3)M$.

Inside the sphere, $r < R$, we have to split the integral up in two:

$$
\Phi_M(r) = -M_0 \cos \theta \frac{\partial}{\partial r} \left[ \frac{1}{r} \int_0^r \frac{r^2}{r_> \, dr'} + \int_r^R \frac{r^2}{r_> \, dr'} \right]
$$

$$
= -M_0 \cos \theta \frac{\partial}{\partial r} \left[ \frac{1}{3} r^2 + \frac{1}{2} (R^2 - r^2) \right]
$$

$$
= \frac{1}{3} M_0 r \cos \theta
$$

$$
= \frac{1}{3} M_0 z.
$$

(9.2.10)

The auxiliary field inside the sphere is just $H = -\frac{1}{3} M_0 \hat{z} = -\frac{1}{3} M$. Outside the sphere, we get

$$
H = -\hat{r} \frac{\partial \Phi_M}{\partial r} - \hat{\theta} \frac{\partial \Phi_M}{\partial \theta} = \frac{2}{3} M_0 \left( \frac{R}{r} \right)^3 \cos \theta \hat{r} + \frac{1}{3} M_0 \left( \frac{R}{r} \right)^3 \sin \theta \hat{\theta}
$$

$$
= \frac{1}{3} M_0 \left( \frac{R}{r} \right)^3 \left( 2 \cos \theta \hat{r} + \sin \theta \hat{\theta} \right)
$$

$$
= \frac{1}{3} M_0 \left( \frac{R}{r} \right)^3 \left( 3 \cos \theta \hat{r} - \hat{z} \right)
$$

$$
= \frac{1}{3} \left( \frac{R}{r} \right)^3 \left[ 3(M \cdot \hat{r}) \hat{r} - M \right].
$$

(9.2.11)

Outside the sphere, the magnetic field is simply $B = \mu_0 H$ since $M = 0$. Inside the sphere, $B = \mu_0 (H + M) = \frac{2}{3} \mu_0 M$. Thus, the magnetic field is

$$
B = \begin{cases} \frac{2}{3} \mu_0 M, & r \leq R, \\ \frac{\mu_0}{3} \left( \frac{R}{r} \right)^3 \left[ 3(M \cdot \hat{r}) \hat{r} - M \right], & r > R. \end{cases}
$$

(9.2.12)
The electric field vanishes inside the sphere, since the charge is glued only to the surface, and is the field of a point charge $Q$ everywhere outside:

$$E = \begin{cases} 
0, & r \leq R, \\
\frac{Q}{4\pi r^2}, & r > R.
\end{cases}$$  \hspace{1cm} (9.2.13)

(b) In part (a), instead of working with $\Phi_M$, we could have worked with the vector potential defined by $B = \nabla \times A$. Then, $\nabla \times H = 0$ implies $\nabla^2 A = \mu_0 \nabla \times M$. Following the same procedure as before, we may write $A$ as

$$A(r) = \frac{\mu_0}{4\pi} \nabla \times \int \frac{M(r')}{|r - r'|} \, d^3 r' = \frac{\mu_0 M_0}{4\pi} \left( \hat{x} \frac{\partial}{\partial x} - \hat{y} \frac{\partial}{\partial y} \right) \int_0^R r'^2 \, dr' \int dS' \frac{1}{|r - r'|} = \frac{\mu_0 M_0}{4\pi} \left( \hat{x} \frac{\partial}{\partial x} - \hat{y} \frac{\partial}{\partial y} \right) \int_0^R r'^2 \, dr'. \hspace{1cm} (9.2.14)$$

Computing this exactly the same way we did before yields

$$A(r) = \frac{1}{3} \mu_0 M r \sin \theta \left\{ \begin{array}{ll} 1, & r \leq R, \\ (R/r)^3, & r > R. \end{array} \right. \hspace{1cm} (9.2.15)$$

This varies with time since $M$ does and thus induces an electric field

$$E_{\text{ind}}(r) = -\dot{A} = -\frac{1}{3} \mu_0 M \dot{r} \sin \theta \dot{\phi} \left\{ \begin{array}{ll} 1, & r \leq R, \\ (R/r)^3, & r > R. \end{array} \right. \hspace{1cm} (9.2.16)$$

A charge $dq = \frac{Q}{4\pi} \sin \theta d\theta d\phi d\phi$ on the surface of the sphere will therefore feel an induced force, which causes an induced torque on the sphere given by

$$d\tau = R\hat{r} \times dq E_{\text{ind}}(R\hat{r}) = \frac{\mu_0 \dot{M} QR^2}{12\pi} \sin^2 \theta \, d\theta \, d\phi \, d\phi. \hspace{1cm} (9.2.17)$$

The $\hat{x}$ and $\hat{y}$ components of $\dot{\hat{r}}$ are multiplied by $\cos \phi$ and $\sin \phi$ respectively, whose integrals over $\phi$ vanish. Hence, we only need the $\hat{z}$ component:

$$d\tau_z = -\frac{\mu_0 \dot{M} QR^2}{12\pi} \sin^3 \theta \, d\theta \, d\phi, \hspace{1cm} (9.2.18)$$

which we integrate to get the total torque

$$\tau = -\frac{\mu_0 \dot{M} QR^2}{12\pi} \int_0^\pi \sin^3 \theta \, d\theta \int_0^{2\pi} d\phi \, \hat{z} = -\frac{2}{9} \frac{\mu_0 \dot{M} QR^2}{12\pi} \hat{z}. \hspace{1cm} (9.2.19)$$

The integral of $\sin^3 \theta$ can easily be computed by replacing two of the sin factors by $1 - \cos^2 \theta$.

We integrate this over all time to find the total angular momentum transferred to the sphere:

$$L = \int_{t_i}^{t_f} \tau \, dt = -\frac{2}{9} \mu_0 [M(t_f) - M(t_i)] \hat{z} = \frac{2}{9} \mu_0 \dot{M} QR^2 \hat{z}. \hspace{1cm} (9.2.20)$$
9.2. ROTATION DUE TO DEMAGNETIZATION

The angular momentum was transferred from the electric and magnetic fields to the sphere itself in order to conserve angular momentum. Let us see how this works in detail.

The angular momentum density in electromagnetic fields is
\[ \ell = \epsilon_0 r \times (E \times B) = \frac{1}{c} \mathbf{r} \times \mathbf{S} \]
where \( \mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} \) is the Poynting vector. By the way, one can essentially write \( \ell \) as
\[ \ell^k \sim \epsilon^{ijk} r^i T^0 j, \]
where \( T^{\mu \nu} \) is the Belinfante (i.e. properly symmetrized) energy momentum tensor of \( E \& M \).

For the purpose of computing the initial angular momentum, we need only consider the region outside the sphere since the electric field vanishes inside the sphere. Furthermore, it will be convenient to use the expression for \( B \) that is purely in spherical coordinates (second line in (9.2.10)).

The angular momentum density is
\[ \ell = \epsilon_0 r \hat{r} \times (E \times B) \]
\[ = -\frac{\mu_0 Q M R^3}{12 \pi r^4} \sin \theta \hat{\theta} \]
\[ = -\frac{\mu_0 Q M R^3}{12 \pi r^4} \sin \theta (\cos \theta \cos \phi \hat{x} + \cos \theta \sin \phi \hat{y} - \sin \theta \hat{z}). \] (9.2.21)

By symmetry, we should expect that the angular momentum exists only about the \( z \)-axis. Therefore, the integral outside the sphere of the \( x \) and \( y \) components of \( \ell \) ought to vanish. It is clear that their \( \phi \) integrals indeed vanish. Therefore, we are left with
\[ \mathbf{L} = \frac{\mu_0 Q M R^3}{12 \pi} \int_R^\infty dr \int_0^\pi \sin^3 \theta \, d\theta \int_0^{2\pi} d\phi \hat{z} = \frac{2}{3} \mu_0 M Q R^2 \hat{z}. \] (9.2.22)

By conservation of angular momentum, this must be the angular momentum of the sphere afterwards since there will no longer be any \( B \) field after \( M \) has vanished. This agrees with 9.2.20.

**Aside 1:** A word on angular momentum conservation: construct the tensor \( M^{\mu \nu} = T^{\mu \nu \rho} - T^{\nu \rho \mu} \) so that \( \epsilon^{ijk} r^i T^0 j \sim M^{0ij} \). Then, \( \partial_\mu T^{\mu \nu} = 0 \), which gives the usual conservation of energy and linear momentum, also implies \( \partial_\mu M^{\mu \nu} = 0 \) which is conservation of angular momentum.

**Aside 2:** It may have occurred to you that since the charged sphere ends up spinning, it will produce a magnetic field at the end (i.e. the magnetic field doesn’t completely vanish, as we supposed.) Thus, there will be some angular momentum remaining in the fields and not all of the angular momentum is transferred to the sphere. You could compute this by computing the magnetic field produced by a spinning charged sphere, the vector potential for which is worked out in Griffiths Example 5.11 (p.236-237). I suppose you could work out how much angular momentum is actually transferred to the sphere, taking this correction into account. However, assuming that the sphere is relatively massive so that it does not end up spinning very quickly, this correction can be expected to be small.

By the way, from the looks of it, this does not seem to have occurred to the professor.
9.3 Quadrupole Lensing

(Spring 2007 Classical (Morning), Problem 6) A quadrupole is a magnet in which the magnetic scalar potential \( B = -\nabla \phi \) is given by \( \phi(x, y, z) = B_0 xy/R \) for \( x^2 + y^2 < R^2 \), where \( R \) is a constant (pole radius) and \( B_0 \) is the field at the pole.

(a) Demonstrate that for relativistic charged particles moving along the \( z \)-axis, the quadrupole acts as a lens. Is the focusing the same in the \( x \) and \( y \) directions?

(b) Consider a “realistic” warm-iron quadrupole of length \( L = 2 \text{ m} \), \( B = 1 \text{ T} \) and \( R = 20 \text{ cm} \). Determine the focal length \( f \) for 20 GeV electrons.

SOLUTION:

(a) Impulse approximation: Since the particle is moving relativistically, we will assume that it takes a negligible amount of time to traverse the effective field region of the quadrupole. Therefore, we will make the impulse approximation that the speed in the \( z \)-direction is unchanged and that we will simply multiply the transverse force experienced by the charges right upon entering the field region by the time it takes to traverse the field region and take that to be the total transverse impulse imparted on the particles. The transverse speeds will be only a small correction to the horizontal speed. Thus, in computing the Lorentz force, we will use the same velocity, \( v = v_0 \hat{z} \). First, let us compute the magnetic field and force:

\[
B = -\frac{B_0}{R} (y \hat{x} + x \hat{y}), \quad F = qv \times B \approx \frac{qv_0 B_0}{R} (x \hat{x} - y \hat{y}). \tag{9.3.1}
\]

The force in the \( x \)-direction is repulsive (tends to increase \( x \)) and in the \( y \)-direction is attractive (tends to decrease \( y \)). Therefore, if we start out with a beam with circular cross-section, the outgoing beam cross-section becomes more and more elliptical. Eventually, it will be focused to a line, and then spread out again. But, as it spreads out again in the \( y \) direction, it gets forced back in, so it oscillates in the \( y \) direction and keeps spreading out in the \( x \) direction. Of course, our assumption is that the beam does not spend nearly enough time to see anywhere near even one oscillation.

![Figure 9.2: Focusing of a beam moving to the right due to quadrupole.](image)

It takes the particle a time \( \Delta t \approx L/v_0 \) to traverse the quadrupole of length \( L \). Suppose that the initial \( x \) and \( y \) position of the incoming particle is \( r_0 = x_0 \hat{x} + y_0 \hat{y} \). Then, the impulse is

\[
\Delta p = F \Delta t = \frac{qLB_0}{R} (x_0 \hat{x} - y_0 \hat{y}). \tag{9.3.2}
\]

The final velocity is \( v_f = v_0 \hat{z} + \Delta p/\gamma m \) and the position is \( r(t) = r_0 + v_f t \). The times it takes for \( r_x(t) \) to vanish in the case \((x_0, y_0) = (R, 0)\) and for \( r_y(t) \) to vanish in the case \((x_0, y_0) = (0, R)\) are

\[
t_x = -\frac{\gamma m R}{qLB_0}, \quad t_y = \frac{\gamma m R}{qLB_0}. \tag{9.3.3}
\]
9.3. QUADRUPOLE LENSING

Of course, the time is negative for the $x$-direction since the beam spreads out in that direction rather than focuses in. The focal lengths are the distances travelled by the particle in these times:

$$f = f_y = -f_x = \frac{\gamma m R v_0}{q L B_0} = \frac{\rho_0 R}{q L B_0},$$  \hspace{1cm} (9.3.4)

where $\rho_0$ is the incoming relativistic momentum. The quadrupole is a diverging lens in the $x$-direction and a converging lens in the $y$-direction and their focal lengths are the same (up to sign).

**Small transverse speed:** We can solve the problem by assuming $\dot{x}/v_0, \dot{y}/v_0 \ll 1$ and $\ddot{z}/v_0 \approx 1$. Now, $\gamma^{-2} = 1 - (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$, where we have set $c = 1$ and you might think that we will be expanding $\gamma$ as $\gamma^{-2} \approx 1 - v_0^2$, but there is no need since $\gamma^{-2} = 1 - v_0^2$ exactly since magnetic fields can change the direction of the charged particles, but not their speed.

Since the particles are relativistic, we must be careful in writing the force equation: $\dot{p} = q v \times B$ still holds, but $p = \gamma m v$, not just $m v$. Plugging this into the force equation yields an equation for each direction:

$$\ddot{x} = \alpha x \dot{z}, \quad \ddot{y} = -\alpha y \dot{z}, \quad \ddot{z} = \alpha (y \dot{x} - x \dot{y}),$$  \hspace{1cm} (9.3.5)

where $\alpha = q B_0 / \gamma m R$. You might like to check that these equations are not independent: they satisfy the relation $\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = v_0^2$ = constant. So, with this one relation, two of the above three equations are independent, and we will choose to use the first two.

Using the constancy of total speed relation, and using the smallness of the transverse speeds, the two equations read

$$\ddot{x} = \alpha v_0 x \sqrt{1 - (\dot{x}/v_0)^2 - (\dot{y}/v_0)^2} \approx \alpha v_0 x, \hspace{1cm} (9.3.6a)$$
$$\ddot{y} = -\alpha v_0 y \sqrt{1 - (\dot{x}/v_0)^2 - (\dot{y}/v_0)^2} \approx -\alpha v_0 y. \hspace{1cm} (9.3.6b)$$

Now, these are simple equations to solve, imposing the initial conditions $x(0) = x_0$ and $y(0) = y_0$:

$$x(t) = x_0 \cosh(t/\tau), \quad y(t) = y_0 \cos(\omega t),$$  \hspace{1cm} (9.3.7)

where $1/\tau^2 = \omega^2 = \alpha v_0 = q v_0 B_0 / \gamma m R$.

Here, we see explicitly the oscillatory motion in the $y$ direction and the exponentially growing behavior in the $x$ direction. We already computed the time it takes to traverse the field region: $\Delta t = L/v_0$. This is a small amount of time, where the smallness condition is $\Delta t/\tau = \omega \Delta t \ll 1$. It follows that $x(t)$ and $y(t)$ given above do not change very much while in the field region: $x(t) \approx x_0$ and $y(t) \approx y_0$. Hence, the impulse approximation.

The outgoing speeds in the $x$ and $y$ directions are

$$v_{fx} = \dot{x}(\Delta t) = (x_0/\tau) \sinh(\Delta t/\tau) \approx x_0 \Delta t/\tau^2 = q L B_0 x_0 / \gamma m R, \hspace{1cm} (9.3.8a)$$
$$v_{fy} = \dot{y}(\Delta t) = -\omega y_0 \sin(\omega \Delta t) \approx -\omega^2 y_0 \Delta t = -q L B_0 y_0 / \gamma m R. \hspace{1cm} (9.3.8b)$$

From this point on, everything follows as in the first method.

(b) We can write the focal lengths in terms of the incoming energy:

$$f = \sqrt{E^2 + (mc^2)^2} \frac{R}{q L B_0 c}.$$

\hspace{1cm} (9.3.9)
Furthermore, since the energy is so much greater than the mass (highly relativistic), we may as well drop the mass term:

\[ f \approx \frac{RE}{qLB_0c} \approx 6.67 \text{ m} \]  \hspace{1cm} (9.3.10)

\textbf{Aside:} Since our approximations above essentially arise from the assumption that the horizontal speed does not change very much, you could presumably Lorentz boost to the horizontal rest frame of the particles. You would need to transform the field tensor appropriately. The new fields will have electric as well as magnetic fields. Since we’re now in the rest frame only electric forces arise. The time it takes to traverse the field region is now smaller because the region is Lorentz contracted. However, you should find that the impulse is unchanged!

When we work in the lab frame, \( \Delta t \) is the time observed in the lab frame, which has no dilation \( \gamma \) factor. Also, there was some confusion about why we divide \( \Delta p_x \) by \( \gamma m \) to get \( v_x \), when \( v_x \) is nonrelativistic. The reason is that \( \mathbf{p} = \gamma m \mathbf{v} \) where \( \gamma \) is the whole factor pertaining to all of \( \mathbf{v} \). So, even if one component is nonrelativistic, the \( \gamma \) factor is still there as long as the total speed is relativistic.
9.4 Magnetic Lensing by a Wire

(Spring 2011 E & M, Problem 1) A cylinder of length $L$ and radius $R$ carries a uniform current $I$ parallel to its axis.

(a) Find the direction and magnitude of the magnetic field inside the cylinder. [Ignore end effects.]

(b) A beam of particles, each with momentum $p_0$ parallel to the cylinder axis and each with positive charge $q$, impinges on its end from the left. Show that after passing through the cylinder the particle beam is focused to a point. [Make a “thin lens” approximation by assuming that the cylinder is much shorter than the focal length. Neglect the slowing down and scattering of the beam particles by the material of the cylinder.] Compute the focal length.

\[ \text{Figure 9.3: Spring 2011 E & M, Problem 1} \]

SOLUTION:

(a) Cylindrical symmetry implies $B = B(r) \hat{\phi}$, where $\hat{\phi}$ is the unit vector that revolves around the wire in the direction that ones four right fingers would do were one to point ones right thumb in the direction of the current, which is taken to be the $+z$ direction. Here, $r$ is the radial direction from the $z$ axis (the cylindrical radial direction). Since the current is evenly spread through the wire, the current density is $J = \frac{I}{\pi R^2} \hat{z}$. Using $\nabla \times B = \mu_0 J$ and Stoke’s theorem,

\[
2\pi r B(r) = \oint B \cdot d\ell = \int (\nabla \times B) \cdot da = \mu_0 \int J \cdot da = \mu_0 \int \left( \frac{r}{R} \right)^2 .
\]

Therefore, the magnetic field magnitude is

\[
B(r) = \frac{\mu_0 I r}{2\pi R^2} .
\]

Writing $\hat{\phi} = -y\hat{x} + x\hat{y}$ allows us to write the magnetic field as

\[
B = \frac{B_0}{R} (-y\hat{x} + x\hat{y}) \quad \text{where} \quad B_0 = \frac{\mu_0 I}{2\pi R} .
\]

(b) Method 1 (Impulse approximation): Since the particle is moving relativistically, we will assume that it takes a negligible amount of time to traverse the effective field region of the quadrupole. Therefore, we will make the impulse approximation that the speed in the $z$ direction is unchanged and that we will simply multiply the transverse force experienced by the charges right upon entering the field region by the time it takes to traverse the field region and take that to be the total transverse impulse imparted on the particles. The transverse speeds will be only a small correction to the horizontal speed. Thus, in computing the Lorentz force, we will use the same velocity, $v = v_0\hat{z}$. The magnetic force is

\[
F = qv \times B \approx -\frac{qv_0 B_0}{R} (x\hat{x} + y\hat{y}) = -\frac{qv_0 B_0}{R} \cdot r .
\]
It takes the particle a time $\Delta t \approx L/v_0$ to traverse the quadrupole of length $L$. Suppose that the initial radial distance is $r_0$. Then, the impulse is

$$\Delta p = F \Delta t = -\frac{qLB_0}{R} r_0. \quad (9.4.5)$$

The final velocity is $v_f = v_0 \hat{z} + \Delta p/\gamma \mu$ and the position is $r(t) = r_0 + v_f t$. The times it takes for $r_x(t)$ to vanish

$$t = \frac{\gamma \mu R}{qLB_0} \quad (9.4.6)$$

Of course, the time is negative for the $x$-direction since the beam spreads out in that direction rather than focuses in. The focal lengths are the distances travelled by the particle in these times:

$$f = \frac{\gamma \mu R v_0}{qLB_0} \quad (9.4.7)$$

where $p_0$ is the incoming relativistic momentum.

Method 2 (Small transverse speed): We assume that $\dot{z}/v_0, \dot{y}/v_0 << 1$ and $\dot{z}/v_0 \approx 1$. Now, $\gamma^{-2} = 1 - (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$, where we have set $c = 1$ and you might think that we will be expanding $\gamma$ as $\gamma^{-2} \approx 1 - v_0^2$, but there is no need since $\gamma^{-2} = 1 - v_0^2$ exactly since magnetic fields can change the direction of the charged particles, but not their speed.

Since the particles are relativistic, we must be careful in writing the force equation: $\dot{p} = qv \times B$ still holds, but $p = \gamma \mu v$, not just $mv$. Plugging this into the force equation yields an equation for each direction:

$$\ddot{x} = -\alpha x \dot{z}, \quad \ddot{y} = -\alpha y \dot{z}, \quad \ddot{z} = \alpha (x \dot{x} + y \dot{y}), \quad (9.4.8)$$

where $\alpha = qB_0/\gamma \mu R$. You might like to check that these equations are not independent: they satisfy the relation $\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = v_0^2$ = constant. So, with this one relation, two of the above three equations are independent, and we will choose to use the first two.

Using the constancy of total speed relation and the smallness of the transverse speeds gives $\dot{z} \approx v_0$. The two equations combine to

$$\ddot{r} \approx -\alpha v_0 r. \quad (9.4.9)$$

Now, this is just an oscillatory equation. Imposing the initial condition $r(0) = r_0$ gives $r(t) = r_0 \cos(\omega t)$, where $1/\tau^2 = \omega^2 = \alpha v_0 = qv_0 B_0/\gamma \mu R$. We already computed the time it takes to traverse the field region: $\Delta t = L/v_0$. This is a small amount of time, where the smallness condition is $\Delta t/\tau = \omega \Delta t << 1$. It follows that $r(t)$ given above does not change very much while in the field region: $r(t) \approx r_0$. Hence, the impulse approximation.

The outgoing speed in the radial direction is

$$(v_r)_f = \dot{r}(\Delta t) = (r_0/\tau) \sin(\omega \Delta t) \approx -\omega^2 r_0 \Delta t = -qLB_0 r_0/\gamma \mu R. \quad (9.4.10)$$

From this point on, everything follows as in the first method.

Method 3 (Relativity): The approximation is that $\dot{z} = v_0 = \text{const}$. Therefore, we may boost to the rest frame of the particles in the beam of charges. In this frame, the charges just sit still, but we will find that there is now an electric field pointing radially inwards forcing
9.4. MAGNETIC LENSING BY A WIRE

the charges towards the central axis. Set \( c = 1 \) so that \( \beta = v_0 \) and let \( \gamma \) be the corresponding gamma-factor. Let the primed frame be the horizontal rest frame of the charges and the unprimed frame the lab. The coordinates are related via

\[
\begin{pmatrix}
  t' \\
  x' \\
  y' \\
  z'
\end{pmatrix} = \begin{pmatrix}
  \gamma & 0 & 0 & -\beta\gamma \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  -\beta\gamma & 0 & 0 & \gamma
\end{pmatrix} \begin{pmatrix}
  t \\
  x \\
  y \\
  z
\end{pmatrix} = \begin{pmatrix}
  \gamma(t - \beta z) \\
  x \\
  y \\
  \gamma(z - \beta t)
\end{pmatrix}. \tag{9.4.11}
\]

Focus on a particle at \( z' = 0 \), then \( z = \beta t \) and thus \( t' = \gamma(t - \beta z) = \gamma(1 - \beta^2)t = t/\gamma \). This yields the usual time dilation relation between the time in the charge rest frame and the time in the lab frame:

\[
t = \gamma t'. \tag{9.4.12}
\]

In the lab frame, the field strength tensor is

\[
F^{\mu\nu} = \begin{pmatrix}
  0 & E_x & E_y & E_z \\
  -E_x & 0 & B_z & -B_y \\
  -E_y & -B_z & 0 & B_z \\
  -E_z & B_y & -B_z & 0
\end{pmatrix} = \frac{B_0}{R} \begin{pmatrix}
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & -x \\
  0 & 0 & 0 & -y \\
  x & y & 0 & 0
\end{pmatrix}. \tag{9.4.13}
\]

In the primed frame, this transforms into

\[
F' = \Lambda F \Lambda^T = \begin{pmatrix}
  \gamma & 0 & 0 & -\beta\gamma \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  -\beta\gamma & 0 & 0 & \gamma
\end{pmatrix} \begin{pmatrix}
  B_0 \\
  0 & 0 & 0 & -x \\
  0 & 0 & 0 & -y \\
  0 & x & y & 0
\end{pmatrix} = \frac{\gamma B_0}{R} \begin{pmatrix}
  0 & -\beta x & -\beta y & 0 \\
  \beta x & 0 & 0 & -x \\
  \beta y & 0 & 0 & -y \\
  0 & x & y & 0
\end{pmatrix}. \tag{9.4.14}
\]

Since \( x = x' \) and \( y = y' \), we could put primes here or not. In the new frame, the electric field is

\[
E' = -\frac{\beta\gamma B_0}{R}(x'\hat{x} + y'\hat{y}) = -\frac{\beta\gamma B_0}{R} \hat{y}'. \tag{9.4.15}
\]

The force is \( F' = qE' = -\beta\gamma \frac{qB_0}{R} \hat{y}' \). This yields an oscillator equation with frequency \( \omega' = [\beta\gamma \frac{qB_0}{mR}]^{1/2} \) and solution \( r' = r_0' \cos \omega' t' \). The contracted length of the cylinder is \( L' = L/\gamma \). Thus, the time for it pass the particle is \( \Delta t' = L'/\beta = L/\beta\gamma \). As expected, the radial position of the particle will hardly have moved in this short time, but it will have developed a radial speed:

\[
r'(\Delta t') \approx r_0', \quad \dot{r}'(\Delta t') \approx \omega^2 r_0' \Delta t' = -\frac{qB_0 L}{mR} x_0'. \tag{9.4.16}
\]

The time it takes to reach the central axis is

\[
t' = \frac{x'(\Delta t')}{|\dot{x}'(\Delta t')|} \approx \frac{mR}{qB_0 L}. \tag{9.4.17}
\]

The corresponding time in the lab frame is just \( t = \gamma t' \), which agrees with Method 1’s Eqn. (9.4.6). Everything else follows as before.
Chapter 10

Electromagnetic Waves

- Wave equations (vacuum or no sources): $\Box \mathbf{E} = \Box \mathbf{B} = 0$ where $\Box = -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \nabla^2$.
- Wave travels in the $\mathbf{E} \times \mathbf{B}$ direction. The amplitudes are related by $E_0 = B_0 c$.
- Polarization: the axis of $\mathbf{E}$ (linear).
- Intensity: average power per unit area $I = \langle S \rangle = \frac{1}{2} \varepsilon_0 E_0^2 c$.
- Radiation pressure: $P = I/c$ (absorbed) and $P = 2I/c$ (reflected).
- Index of refraction: speed of light in medium is $v = 1/\sqrt{\mu n}$, This gives $n = \sqrt{\varepsilon \mu/\varepsilon_0 \mu_0}$. Since $\mu$ does not vary much in normal materials, usually $n \approx \sqrt{\kappa}$, where $\kappa = 1 + \chi_c$ is the dielectric constant.
- Boundary conditions for linear material and no interfacial free charge or current:
  \[ \epsilon_1 E_{1z} = \epsilon_2 E_{2z}, \quad B_{1z} = B_{2z}, \quad E_{1||} = E_{2||}, \quad \frac{1}{\mu_1} B_{1||} = \frac{1}{\mu_2} B_{2||}. \]
- Normal incidence (and $\mu \approx \mu_0$): $R = \left( \frac{n_1 - n_2}{n_1 + n_2} \right)^2$ and $T = \frac{4n_1 n_2}{(n_1 + n_2)^2}$.
- $r_{||} = \left( \frac{E_{||0}}{E_{||0}} \right)_{||} = \frac{\alpha_{||}}{\alpha_{||} + \beta_{||}}$ and $r_{\perp} = \left( \frac{E_{\perp0}}{E_{\perp0}} \right)_{\perp} = \frac{1 - \alpha_{\perp}}{1 + \alpha_{\perp}}$ and $t_{||} = \left( \frac{E_{||0}}{E_{||0}} \right)_{||} = \frac{2}{\alpha_{||} + \beta_{||}}$ and $t_{\perp} = \left( \frac{E_{\perp0}}{E_{\perp0}} \right)_{\perp} = \frac{2}{1 + \alpha_{\perp}}$ are the Fresnel equations where $\alpha = \cos \theta_T/\cos \theta_I$ and $\beta = \mu_1 n_2/\mu_2 n_1$. Then, $R = r^2$ and $T = t^2$.
- Stoke’s relations: let $r$ and $t$ be the amplitude reflection and transmission coefficients going from medium 1 to medium 2, and $r'$ and $t'$ from 2 to 1. Then, $r = -r'$ and $t^* t + r^2 = 1$.
- Wave equations in conductors: $\Box \mathbf{E} = \mu \sigma \mathbf{E}$ and $\Box \mathbf{B} = \mu \sigma \mathbf{B}$. Admit plane waves, but with dispersion $(k v)^2 = \omega^2 + i(\sigma/\epsilon) \omega$, where $c^2 = 1/\mu \epsilon$.
- Skin depth: $d = 1/\kappa$ where $\kappa = \text{Im } \tilde{k}$.
- Diffraction: the slit function $f(X)$ is 1 at points $X = (X, Y)$ on the diffracting screen where light may pass and is 0 elsewhere. Let $x = (x, y)$ denote the points on the observation screen a large distance $L$ away. The normalized intensity on the faraway screen is $I(q) = I_0 |G(q)|^2$ where $G(q) = \int \int \int (f(X)) e^{-iq.X} d^2 X$ and $f(X) d^2 X$ and where $q = k_0 X$. Here $k = 2\pi/\lambda$ is the wavevector and $I_0$ is the intensity incident on the diffracting screen.
CHAPTER 10. ELECTROMAGNETIC WAVES

10.1 E&M Wave in a Metal

[Fall 2008 Classical (Morning), Problem 1] Consider a metal with a density \( n \) of conduction electrons that are free to move without damping. Each electron has mass \( m \) and charge \(-e\).

Suppose that an electromagnetic plane wave is sent into the metal with frequency \( \omega \).

(a) Derive an expression that shows how the wave vector \( k \) of the plane wave depends on frequency \( \omega \).

(b) There is a threshold frequency below which plane waves cannot travel through this material. Derive an expression for this frequency.

SOLUTION:

(a) For the moment, suppose that the free electrons feel a damping force proportional to their velocity, parametrized by \( \gamma \), which we will later set to zero. Suppose an electric field, \( \mathbf{E} = E_0 e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)} \) is incident. Then, the force equation for the motion of an electron is

\[
-m \ddot{x} + m \dot{x} = -e \mathbf{E}.
\]

Posit the ansatz \( x = x_0 e^{i (\mathbf{k} \cdot \mathbf{x}_0 - \omega t)} \), where we have assumed \( x \) to not stray very far away from its mean value, \( x_0 \) (hence the position in the exponential is \( x_0 \), not \( x \)). Another way of saying this is that we are assuming that the electric field does not vary appreciably within the vicinity of a single free electron. Then, the force equation reads

\[
-m(\omega^2 + i\omega\gamma)x = -e E_0 \implies x = \frac{e E_0 / m}{\omega^2 + i\omega\gamma}.
\] (10.1.1)

It follows from \( \mathbf{P} = -n e \mathbf{x} \) that

\[
\mathbf{P} = -\frac{ne^2}{m(\omega^2 + i\omega\gamma)} \mathbf{E} \implies \epsilon(\omega) = \epsilon_0 \left( 1 - \frac{ne^2}{\epsilon_0 m(\omega^2 + i\omega\gamma)} \right).
\] (10.1.2)

Define the plasma frequency \( \omega_p^2 = ne^2 / \epsilon_0 m \). Then, we write the permittivity as

\[
\epsilon(\omega) = \epsilon_0 \left( 1 - \frac{\omega_p^2}{\omega^2 + i\omega\gamma} \right).
\] (10.1.3)

We will assume that \( \epsilon(\omega) \) varies only by a small amount from \( \epsilon_0 \). This implies that the term involving the plasma frequency is assumed to be sufficiently small.

Next, from \( \mathbf{J} = -n e \dot{x} \) we have

\[
\mathbf{J} = \frac{ine^2}{m(\omega + i\gamma)} \mathbf{E} \implies \sigma(\omega) = \frac{i\epsilon_0 \omega_p^2}{\omega + i\gamma}.
\] (10.1.4)

The wave equation in a conductor reads \( \nabla \mathbf{E} = \mu \sigma \dot{\mathbf{E}} \), which, upon inserting the form \( \mathbf{E} = \mathbf{E}_0 e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)} \) reads (Griffiths 9.124, p.394):

\[
\mathbf{k}^2 = \mu e \omega^2 + i \mu \sigma \omega = (\mu e + i \mu \sigma / \omega) \omega^2.
\] (10.1.5)

Writing \( \mathbf{k} = k + i\kappa \), we find (Griffiths 9.126, p.394):

\[
k = \omega \sqrt{\frac{\epsilon_0 \mu}{2}} \left[ \sqrt{1 + \left( \frac{\sigma}{\epsilon_0 \omega} \right)^2} + 1 \right]^{1/2}.
\] (10.1.6)
10.1. E&M WAVE IN A METAL

However, note that
\[ \frac{\sigma}{\epsilon \omega} = \frac{i \omega_p^2}{\omega^2 + i \omega \gamma} \left[ 1 - \frac{\omega_p^2}{\omega^2 + i \omega \gamma} \right]^{-1}, \] (10.1.7)

which is at least linear in the quantity that measures the deviation of \( \epsilon(\omega) \) from \( \epsilon_0 \), which we have assumed to be small. Thus,
\[ k = \omega \sqrt{\epsilon \mu} \left[ 1 + \mathcal{O}(1 - (\epsilon/\epsilon_0)^2) \right]. \] (10.1.8)

We assume that \( \mu \approx \mu_0 \) since permeability really doesn’t vary much for most materials. Then, the dispersion reads
\[ k = \omega \sqrt{\mu_0 \epsilon_0} \left( 1 - \frac{(\omega_p/\omega)^2}{1 + i(\gamma/\omega)} \right)^{1/2} \xrightarrow{\gamma \to 0} \frac{\omega}{c} \sqrt{1 - (\omega_p/\omega)^2}. \] (10.1.9)

(b) Requiring that \( k \) be real implies
\[ \omega > \omega_p = \frac{\sqrt{n^2 c^2 / \epsilon_0 \mu_0}}{m}. \] (10.1.10)

Aside: It would probably have bit a bit neater to just set \( \gamma = 0 \) from the start. Also, generically, far away from any resonances and when damping is negligible, the dispersion does simply take the form \( \omega = kc' \) where \( c' \) is the speed of light in the medium (i.e. \( c' = c/n \), where \( n \) is the index of refraction of the medium). As in the professor’s solution, you could have just started from there. Then you would only need the frequency dependence of the permittivity, which we found in (10.1.3).
10.2 Four Displaced Sources

[Lim 4007] Four identical coherent monochromatic wave sources A, B, C, D, as shown in the diagram, produce waves of the same wavelength, \( \lambda \). Two receivers, \( R_1 \) and \( R_2 \), are at great (but equal) distances from B.

(a) Which receiver picks up the greater signal?

(b) Which receiver, if any, picks up the greater signal if source B is turned off?

(c) What about if source D is turned off?

(d) Which receiver can tell which source, B or D, has been turned off?

SOLUTION:

(a) Let \( r \) be the distance of \( R_1 \) and \( R_2 \) from B such that \( r \gg \lambda \). The ratio of the amplitudes of the electric fields at the two detectors with the amplitudes of each source are (neglecting \( 1/r^2 \) and \( e^{iwt} \) terms common to all sources):

\[
E_1/E_0 = e^{ik(r-\lambda/2)} + e^{ikr} + e^{ik(r+\lambda/2)} + \exp[ik(r^2 + (\lambda/2)^2)^{1/2}], \quad (10.2.1a)
\]

\[
E_2/E_0 = e^{ikr} + e^{ik(r+\lambda/2)} + 2 \exp[ik(r^2 + (\lambda/2)^2)^{1/2}], \quad (10.2.1b)
\]

Using \( k = 2\pi/\lambda \), we find \( e^{\pm i\lambda/2} = e^{\pm i\pi} = -1 \). Furthermore, we make the approximation \((r^2 + (\lambda/2)^2)^{1/2} \approx r \) since \( r \gg \lambda \). Then, the received amplitudes become

\[
E_1/E_0 \approx -e^{ikr} + e^{ikr} - e^{ikr} + e^{ikr} = 0, \quad (10.2.2a)
\]

\[
E_2/E_0 = e^{ikr} - e^{ikr} + 2e^{ikr} = 2e^{ikr}. \quad (10.2.2b)
\]

Thus, receiver two measures the stronger signal. In fact, receiver measures zero:

\[
I_1 = 0, \quad I_2 \approx 4E_0^2. \quad (10.2.3)
\]

(b) From the above results, if we turn off B, we have

\[
E_1 \approx -E_0 e^{ikr}, \quad E_2 \approx E_0 e^{ikr}. \quad (10.2.4)
\]

Thus, their measured intensities are equal:

\[
I_1 = I_2 \approx E_0^2.
\]
(c) Now, if source D is turned off, then we have

\[ E_1 \approx -E_0 e^{ikr}, \quad E_2 \approx 3E_0 e^{ikr}. \]  \hspace{1cm} (10.2.5)

Thus, receiver 2 measures the stronger signal:

\[ I_1 \approx E_0^2, \quad I_2 \approx 9E_0^2. \]  \hspace{1cm} (10.2.6)

(d) The signal intensity does not change for receiver 1 whereas it does for receiver 2 when we switch from B off and D on to B on and D off. Receiver 1 can only tell whether or not one or the other is off, but it cannot tell which one. Only receiver 2 can determine which one is off since its signal intensity changes.
10.3 Self-Trapped Beam

[Lim 4017] Beams of electromagnetic radiation eventually spread because of diffraction, among other things (e.g. scattering). Recall that a beam which propagates through a circular aperture of diameter $D$ spreads with a diffraction angle $\theta_d = 1.22 \lambda_n / d$, where $\lambda$ is the wavelength in a material with index of refraction $n$. In many dielectric media, the index of refraction increases in large electric fields and can be well represented by $n = n_0 + n_2 E^2$, where $n_2$ is some positive dimensionfull constant.

Show that in such a nonlinear medium, the diffraction of the beam can be counterbalanced by total internal reflection of the radiation at the boundaries of the beam thus forming a self-trapped beam. Calculate the threshold power for the existence of a self-trapped beam. Assume the radiation to be plane waves.

Figure 10.2: Total internal reflection (right) can compensate for diffraction (left).

SOLUTION:

In the region of the beam, the index of refraction is $n = n_0 + n_2 E^2$ while it is $n_0$ outside, so total internal reflection at the boundary of the beam is possible. Let the angle that a ray incident on the beam boundary makes with the normal to the boundary be $\pi/2 - \theta$, so that $\theta$ is the angle relative to the the boundary surface itself. Then, total internal reflection occurs if $n \sin(\pi/2 - \theta) = n \cos \theta \geq n_0$, or $n \geq n_0 / \cos \theta$. If $\theta_d$ is the Rayleigh diffraction angle, as drawn, then this ray hits the beam boundary at an angle $\theta_d/2$ with respect to the boundary surface itself. Thus, we require

$$n = n_0 + n_2 E^2 \geq \frac{n_0}{\cos(\theta_d/2)} \implies E \geq \sqrt{\frac{n_0}{n_2}} \left( \frac{1}{\cos(\theta_d/2)} - 1 \right).$$

(10.3.1)

For plane waves, the intensity (average Poynting vector) is

$$I = \langle S \rangle = \frac{1}{2} \epsilon E^2 c_n = \frac{1}{2} \sqrt{\frac{\epsilon}{\mu}} E^2,$$

(10.3.2)

where $c_n = 1/\sqrt{\mu} \epsilon$ is the speed of light in the medium.

The average power is just the intensity multiplied by the beam area:

$$P = I \pi D^2 / 4 \geq \frac{\pi D^2}{4} \cdot \frac{1}{2} \sqrt{\frac{\epsilon}{\mu}} \frac{n_0}{n_2} \left( \frac{1}{\cos(\theta_d/2)} - 1 \right).$$

(10.3.3)

At threshold, equality holds and $n = n_0 / \cos(\theta_d/2)$. We can also relate $n$ with the permittivity and permeability: $c_n = 1/\sqrt{\mu} \epsilon \approx 1/\sqrt{\mu_0 \epsilon_0 (\epsilon/\epsilon_0)} = c/\sqrt{\epsilon/\epsilon_0}$ and so $n = \sqrt{\epsilon/\epsilon_0}$. Finally, we write $\sqrt{\epsilon_0/\mu_0} = \epsilon_0 c$, where $c$ is the speed of light in vacuum. Then, we have the threshold or critical power:

$$P_c = \frac{\pi \epsilon_0 c^2 D^2}{8} \frac{n_0^2}{n_2} \frac{1 - \cos(\theta_d/2)}{\cos^2(\theta_d/2)}.$$
Next, we expand the cosines and replace $\theta_d = 1.22\lambda_n / D$:

$$P_c \approx \frac{\pi \epsilon_0 c D^2}{8} \frac{n_0^2}{n_2} \left(\frac{\theta_d}{2}\right)^2 = \frac{\pi \epsilon_0 c n_0^2}{64} \frac{1}{n_2} (1.22\lambda_n)^2.$$ (10.3.5)

Finally, we recognize $n_0\lambda_n$ as the wavelength, $\lambda$, in vacuum. Then,

$$P_c \approx \frac{\pi \epsilon_0 c}{64n_2} (1.22\lambda)^2.$$ (10.3.6)
10.4 Reflection Phase Shift

[Lim 4014] A plane polarized electromagnetic wave travelling in a dielectric medium of refractive index $n$ is reflected at normal incidence from the surface of a conductor. Find the phase change undergone by its electric vector if the refractive index of the conductor is $n_2 = n(1 + i\rho)$.

**SOLUTION:**

Let $\hat{z}$ be normal to the interface and point from region 1 to region 2. The incident, reflected and transmitted waves are

$$E_i = E_{i0}e^{ik(z-c_n t)}, \quad E_r = E_{r0}e^{-ik(z+c_n t)}, \quad E_t = E_{t0}e^{ik'(z-c'_n t)}, \quad (10.4.1)$$

where $c_n = c/n$ and $c'_n = c/n_2$ are the speeds of the light in the two regions. The magnetic fields have $B_{i0} = (n/c)E_{i0}$, $B_{r0} = (n/c)E_{r0}$ and $B_{t0} = (n_2/c)E_{t0}$. A priori, we will take the reflected and incident electric fields to be in the same direction and thus the corresponding magnetic fields to be oppositely oriented.

For any boundary, the component of the electric field parallel to the surface must be continuous. Hence,

$$E_{i0} + E_{r0} = E_{t0}. \quad (10.4.2)$$

Assuming no free currents at the interface, the parallel components of $\mathbf{H}$ must also be continuous:

$$\frac{1}{\mu_1}(B_{i0} - B_{r0}) = \frac{1}{\mu_2}B_{t0}. \quad (10.4.3)$$

The minus sign in the reflected term is due to the fact that we chose a plus sign in (10.4.2). We set $\mu_1 \approx \mu_2 \approx \mu_0$ and use the relations with the electric field amplitudes to write

$$\frac{\mu}{\epsilon}(E_{i0} - E_{r0}) = \frac{\mu}{\epsilon}E_{t0}. \quad (10.4.4)$$

Combining this with (10.4.2) yields

$$E_{r0} = \left(\frac{n - n_2}{n + n_2}\right)E_{i0} = -\frac{i\rho}{2n + i\rho}E_{i0} = \frac{\rho}{\sqrt{\rho^2 + 4}}e^{i(\pi + \varphi)}E_{i0}. \quad (10.4.5)$$

where $\tan \varphi = 2/\rho$.

Thus, the total phase shift is

$$\text{phase shift} = \pi + \tan^{-1}(2/\rho). \quad (10.4.6)$$

This reproduces the fact that, if $\rho = 0$ (real index of refraction), we just get a half-wavelength phase shift.
10.5 Rectangular Waveguide

[Spring 2000 Classical (Afternoon), Problem 6; Griffiths 9.5.2] A rectangular waveguide made of a perfect metal sheet extends from \( x = 0 \) to \( x = a \), \( y = 0 \) to \( y = b \) (with \( a > b \)) and is infinite in the \( z \) direction. For an electromagnetic wave of frequency \( \omega \) propagating along the \( z \)-axis,

(a) find \( \mathbf{E} \) inside the waveguide \( (\epsilon = \mu = 1) \), and the dispersion relation,

(b) find the phase and group velocities \( v_p \) and \( v_g \), and show \( v_g < c < v_p \), but \( v_p \sqrt{\epsilon} = c \),

(c) show that there is a cutoff frequency below which no electromagnetic wave can propagate through the waveguide.

SOLUTION:

(a) A priori, there are six components in the \( \mathbf{E} \) and \( \mathbf{B} \) fields. TE (transverse electric) waves have \( E_z = 0 \) and TM (transverse magnetic) waves have \( B_z = 0 \). Thus, the transversality condition gets rid of one component and we are left with five, whose inter-relations we must work out via the massless Klein-Gordon equation which they satisfy \( (\Box \mathbf{E} = \Box \mathbf{B} = 0) \) and via the boundary conditions \( E_{\parallel} = B_{\perp} = 0 \) at the edges of the waveguide.

On the other hand, we could work with the potential instead, which has four independent components. We will find that the transversality condition eliminates two components and leaves us with two with which to work! That is quite a large simplification compared to the fields case. The price we pay is that we have to deal with the gauge freedom: \( A^\mu \rightarrow A^\mu + \partial^\mu \Lambda \), where \( \Lambda \) is an arbitrary scalar function.

I advocated the second method in class. However, since we are not asked to compute magnetic fields, at least for TE modes, we only have to find the two components, \( E_x \) and \( E_y \), anyway, so the first method actually seems more straightforward in this case. For TM modes, we need to find all three electric field components versus two vector potential components, but the simplification is certainly not as drastic as 5 versus 2.

**Fields method:** Since \( \mathbf{E} \) satisfies \( -\frac{1}{c^2} \Delta \mathbf{E} + \nabla^2 \mathbf{E} = 0 \), it can be written as a superposition of plane waves \( \sim e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \).

For a TE mode, \( E_z = 0 \). The boundary condition, \( E_{\parallel} = 0 \), implies that \( E_x \) vanishes at \( y = 0 \) and \( y = b \) and \( E_y \) vanishes at \( x = 0 \) and \( x = a \). This implies \( k_x = n_x \pi / a \) and \( k_y = n_y \pi / b \) for non-negative integers \( n_x, n_y \) (not both of which vanish). Finally, imposing the Gauss’ law constraint, \( \nabla \cdot \mathbf{E} = 0 \), gives the unique solution

\[
E_x = E_{x0} \cos \left( \frac{n_x \pi x}{a} \right) \sin \left( \frac{n_y \pi y}{b} \right) e^{i(k_z z - \omega t)},
\]

\[
E_y = E_{y0} \sin \left( \frac{n_x \pi x}{a} \right) \cos \left( \frac{n_y \pi y}{b} \right) e^{i(k_z z - \omega t)},
\]

where the amplitudes are related via

\[
E_{x0} \frac{n_x}{a} + E_{y0} \frac{n_y}{b} = 0.
\]

Thus, we may write the fields as

\[
E_x = E_0 \cos \left( \frac{n_x \pi x}{a} \right) \sin \left( \frac{n_y \pi y}{b} \right) e^{i(k_z z - \omega t)},
\]

\[
E_y = -E_0 \frac{n_y}{n_x} \sin \left( \frac{n_x \pi x}{a} \right) \cos \left( \frac{n_y \pi y}{b} \right) e^{i(k_z z - \omega t)}.
\]

(10.5.3)
For a TM mode, the solutions for $E_x$ and $E_y$ with arbitrary coefficients still hold. Since $E_z$ must vanish at all boundaries, its solution is

$$E_z = E_{z0} \sin \left( \frac{n_x \pi x}{a} \right) \sin \left( \frac{n_y \pi y}{b} \right) e^{i(k_z z - \omega t)}. \tag{10.5.4}$$

Now, the condition $\nabla \cdot \mathbf{E} = 0$ implies

$$-E_{z0} \frac{n_x \pi}{a} - E_{y0} \frac{n_y \pi}{b} + ik_z E_{z0} = 0. \tag{10.5.5}$$

The dispersion relation for either mode is

$$\omega^2 = \left[ k_z^2 + (m \pi/a)^2 \right] + \left[ (n \pi/b)^2 \right] c^2 = k_z^2 c^2 + \omega_{mn}^2, \tag{10.5.6}$$

where the above serves as definition for $\omega_{mn}$.

Using this dispersion relation, we may satisfy (10.5.5) by

$$E_x = E_0 \cos \left( \frac{n_x \pi x}{a} \right) \sin \left( \frac{n_y \pi y}{b} \right) e^{i(k_z z - \omega t)},$$

$$E_y = E_0 \frac{n_y}{n_x} \sin \left( \frac{n_x \pi x}{a} \right) \cos \left( \frac{n_y \pi y}{b} \right) e^{i(k_z z - \omega t)}, \tag{10.5.7}$$

$$E_z = -\frac{i \omega^2 - k_z^2 c^2}{m} \sin \left( \frac{n_x \pi x}{a} \right) \sin \left( \frac{n_y \pi y}{b} \right) e^{i(k_z z - \omega t)}.$$  

**Potential method:** The 4-vector potential also satisfies the massless Klein-Gordon equation, and is thus a superposition of $\sim e^{i(k_z z - \omega t)}$.

For a TE mode, $E_z = -\frac{\partial A_y}{\partial t} - \frac{\partial A_z}{\partial y} = i \omega A_z - ik_z V = 0$. Gauge invariance can change $A_z$ and $V$ independently and thus each must separately vanish: $A_z = 0$ and $V = 0$. Therefore, $E_x = i \omega A_x$ and similarly for $E_y$. As in the first method, the boundary conditions imply $k_x = n_x \pi/a$ and $k_y = n_y \pi/b$. Since $V = 0$, the Coulomb gauge ($\nabla \cdot \mathbf{A} = 0$) and the Lorenz gauge ($\nabla \cdot \mathbf{A} = -\frac{1}{c^2} \frac{\partial V}{\partial t}$) are actually equivalent. All of this implies

$$A_x = A_{x0} \cos \left( \frac{n_x \pi x}{a} \right) \sin \left( \frac{n_y \pi y}{b} \right), \quad A_y = A_{y0} \sin \left( \frac{n_x \pi x}{a} \right) \cos \left( \frac{n_y \pi y}{b} \right). \tag{10.5.8}$$

along with the relation

$$A_{x0} \frac{n_x}{a} + A_{y0} \frac{n_y}{b} = 0. \tag{10.5.9}$$

The electric field components are just $E_x = i \omega A_x e^{i(k_z z - \omega t)}$ and similarly for $E_y$. This agrees with (10.5.3) after the identifications $E_{x0} = i \omega A_{x0}$ and $E_{y0} = i \omega A_{y0}$.

For a TM mode, $B_z = \frac{\partial A_x}{\partial x} - \frac{\partial A_y}{\partial y} = -ik_x A_y + ik_y A_x$ and, as before, $A_x = A_y = 0$. Then, $B_x = -ik_y A_z$ must vanish at $x = 0$ and $x = a$ and $B_y = ik_x A_z$ must vanish at $y = 0$ and $y = b$. This implies $k_x = n_x \pi/a$ and $k_y = n_y \pi/b$ and

$$A_z = A_0 \sin \left( \frac{n_x \pi x}{a} \right) \sin \left( \frac{n_y \pi y}{b} \right). \tag{10.5.10}$$

The Lorenz gauge condition now reads $-ik_z A_z = -\frac{i \omega}{c^2} V$ or

$$V = \frac{k_z c^2}{\omega} A_z. \tag{10.5.11}$$

Then, since $A_x = A_y = 0$, we have $E_x = -\frac{\partial V}{\partial x}$ and similarly for $E_y$. Now, $E_z = i \omega A_z - ik_z V$. These are found to be

$$E_x = -\frac{m \pi k_x c^2}{a \omega} A_0 \cos \left( \frac{m \pi x}{a} \right) \sin \left( \frac{n_y \pi y}{b} \right) e^{i(k_z z - \omega t)},$$

$$E_y = -\frac{n \pi k_y c^2}{b \omega} A_0 \sin \left( \frac{m \pi x}{a} \right) \cos \left( \frac{n_y \pi y}{b} \right) e^{i(k_z z - \omega t)}, \tag{10.5.12}$$

$$E_z = \frac{i \omega^2 - k_z^2 c^2}{a \omega} A_0 \sin \left( \frac{m \pi x}{a} \right) \sin \left( \frac{n_y \pi y}{b} \right) e^{i(k_z z - \omega t)}.$$  

Again, with the identification $-\frac{m \pi k_x c^2}{a \omega} A_0 = E_{x0}$, and so on, this is the same result as (10.5.7).
10.5. RECTANGULAR WAVEGUIDE

(b) The phase velocity is

\[ v_p = \frac{\omega}{k_z} = c\sqrt{1 + (\omega_{mn}/k_z c)^2} > c. \]  

(10.5.13)

The group velocity is

\[ v_g = \frac{d\omega}{dk_z} = \frac{c}{\sqrt{1 + (\omega_{mn}/k_z c)^2}} < c. \]  

(10.5.14)

Indeed, they do satisfy \( v_p v_g = c^2 \).

(c) From the dispersion relation, \( \omega > \omega_{mn} \) and, since \( a > b \), the lowest possible value, or the cutoff frequency, is \( \omega > \omega_0 = \pi c/a. \)
10.6 Peculiar Double-Slit Diffraction

[Fall 1995 Classical (Morning), Problem 5] An interference pattern is produced by laser light (wavelength $\lambda$) passing through two slits. One slit has width $w_1 = 10\lambda$ and the other is $w_2 = 20\lambda$ wide, and the slits are separated by $d = 1000\lambda$. The pattern is observed on a screen a large distance $L$ away.

(a) What is the separation $\delta y$ between adjacent interference maxima (neglecting the slit widths)?

(b) What are the widths ($\Delta y_1$ and $\Delta y_2$) of the central maxima of the diffraction patterns due to each slit independently?

(c) How many interference fringes are contained in the central max?

(d) What is the ratio of intensity maxima to intensity minima near the center of the pattern?

SOLUTION:

(a) Let $y$ denote the vertical position on the observation screen with $y = 0$ being the point directly across from the midpoint between the two slits and define $\theta$ such that $\tan \theta = y/L$. Since $L$ is large, we can make the small angle approximation $\sin \theta \approx \tan \theta = y/L$. The extra optical path length travelled by the light ray from the bottom slit in reaching the point $y$ is $d\sin \theta$. Thus, if this equals an integer number of wavelengths, we get constructive interference and if it equals a half-odd integer number of wavelengths, then we get destructive interference. Thus, the $y$-positions of the maxima are $y_n = n\lambda L/d$ and the separation between adjacent maxima is $\delta y = \lambda L/d = L/1000$.

(b) An easy, though not particularly convincing, argument for determining the destructive interference condition for single-slit diffraction is that the ray from the edge of the slit should cancel with the ray essentially at the center of the slit. This gives the destructive interference condition $\frac{\Delta y_2}{2} \sin \theta = \frac{\Delta y_1}{2}$ and thus $y/L = \sin \theta = n\lambda/w$ where $n = 1, 2, \ldots$. The width of the central max is the distance between the $n = \pm 1$ minima, which is $\Delta y = 2\lambda L/w$. Hence, $\Delta y_1 = L/5$ and $\Delta y_2 = L/10$.

(c) The interference pattern for slit 2 shows that the electric field for the signal from slit 2 vanishes at $y = L/20$. For $y \gtrsim L/20$, the electric fields from the two slits will have opposite sign and can cancel soon thereafter. Thus, we will estimate that the width of the combined envelope central max is $2(L/20) = L/10$ in which there are 100 interference fringes of width $\delta y = L/1000$.

(d) Since slit 2 is twice as wide as slit 1 and we are near the center of the central max (and thus we can neglect the interference effects of the widths), the only effect of the widths is that slit 2, treated as a source, is four times more intense than slit 1 (twice as much electric field, or $E_2 = 2E_1$). Maximum occurs when the amplitudes add and minimum occurs when they subtract. Thus, $\left(\frac{E_{\text{max}}}{E_{\text{min}}}\right)^2 = \frac{(E_1 + E_2)^2}{(E_1 - E_2)^2} = \frac{(3E_1)^2}{(-E_1)^2} = 9$.

**Fourier method:** It is fairly easy to use the method of Fourier transforms to solve this problem. For the moment, let the slits have length $b \gg d$, but still satisfying $b << L$. The slit function is

$$f(X,Y) = \begin{cases} 1 & |X| \leq \frac{b}{2} \text{ and } (|Y + \frac{d}{2}| \leq \frac{w_2}{2} \text{ or } |Y - \frac{d}{2}| \leq \frac{w_1}{2}) \\ 0 & \text{otherwise.} \end{cases}$$

(10.6.1)
10.6. PECULIAR DOUBLE-SLIT DIFFRACTION

Its integral is just \( \int dX \int dY f(X,Y) = b(w_1 + w_2) \). The Fourier transform is

\[
\int dX \int dY f(X,Y) e^{-\frac{2\pi i}{\lambda}(xX+yY)} = \int_{-b/2}^{b/2} dX e^{-\frac{2\pi i}{\lambda}xX} \left[ \int_{-d/2}^{d/2} \frac{dx}{\pi} + \int_{-d/2}^{d/2} \frac{dx}{\pi} \right] dY e^{-\frac{2\pi i}{\lambda}yY} = \frac{M}{\pi} \sin \left( \frac{\pi b}{\lambda L} x \right) \frac{M}{\pi} \left[ e^{i\frac{\pi d}{\lambda L} y} \sin \left( \frac{\pi w_1}{\lambda L} y \right) + e^{-i\frac{\pi d}{\lambda L} y} \sin \left( \frac{\pi w_2}{\lambda L} y \right) \right].
\]

(10.6.2)

Thus, the transfer function is

\[
G(x,y) = \frac{\lambda L}{\pi \hbar \kappa} \sin \left( \frac{\pi b}{\lambda L} x \right) \frac{\lambda L}{\pi (w_1 + w_2) \kappa} \left[ e^{i\frac{\pi d}{\lambda L} y} \sin \left( \frac{\pi w_1}{\lambda L} y \right) + e^{-i\frac{\pi d}{\lambda L} y} \sin \left( \frac{\pi w_2}{\lambda L} y \right) \right].
\]

(10.6.3)

The observed intensity is \( I(x,y) = I_0 |G(x,y)|^2 \). In the large \( b \) limit, \( \left( \frac{\lambda L}{\pi \hbar \kappa} \right)^2 \sin^2 \left( \frac{\pi b}{\lambda L} \right) \) approaches the function that vanishes everywhere except at \( x = 0 \) at which point its value is 1. So, we may just neglect the \( x \) dependence altogether:

\[
I(y) = I_0 \left( \frac{\lambda L}{\pi (w_1 + w_2) \kappa} \right)^2 \left[ \sin^2 \left( \frac{\pi w_1 y}{\lambda L} \right) + \sin^2 \left( \frac{\pi w_2 y}{\lambda L} \right) + 2 \cos \left( \frac{2\pi d y}{\lambda L} \right) \sin \left( \frac{\pi w_1 y}{\lambda L} \right) \sin \left( \frac{\pi w_2 y}{\lambda L} \right) \right].
\]

(10.6.4)

Plugging in \( d = 1000\lambda \), \( w_1 = 10\lambda \) and \( w_2 = 20\lambda \) and defining \( \eta = 10\pi y/L \) gives

\[
I(\eta) = \frac{I_0}{\sin^2 \left( \frac{\pi}{2} \right)} \left[ \sin^2 \eta + \sin^2 (2\eta) + 2 \cos(200\eta) \sin \eta \sin(2\eta) \right].
\]

(10.6.5)

From the plot below, it is clear that the first minimum of the outer envelope occurs at \( y = \pm L/20 \), which agrees with our estimate in part (c). We can also see that near \( y = 0 \), the minimum intensity is around \( 1/9 = 0.11 \), and the maximum is 1, which agrees with part (d).

Figure 10.3: Fall 1995 Classical (Morning), Problem 5
10.7 Double-Square Diffraction

(Spring 2004 Classical (Afternoon), Problem 5) Two square slits of dimension $a$ and $3a$ are arranged along a horizontal line $L$ on an opaque screen. The separation between the edges of the two slits is equal to $2a$. The slits are illuminated with collimated monochromatic light of wavelength $\lambda$ and uniform intensity $I_0$ coming from infinity. A lens is placed on the right hand side of the screen to collect the diffracted light and focus the diffracted light onto a screen for viewing the diffraction pattern.

(a) Calculate the intensity distribution of the diffraction pattern due to the slit of dimension $a$ alone (as a function of the coordinates of the diffracted beam on the screen. Choose the origin of the coordinate system appropriately to simplify your answer!)

(b) Do the same, but now for both slits.

Figure 10.4: Spring 2004 Classical (Afternoon), Problem 5

SOLUTION:

(a) Neglect the lens. All it does is justify the usual small angle approximation that parallel rays leaving the slits focus at a point on the observation screen.

Let $\vec{z}$ be the direction from the diffracting screen to the observation screen and let $z$ be their separation. Let $(X,Y)$ be coordinates on the diffracting screen and let $(x, y)$ be coordinates on the observation screen. The general result is as follows: define the slit function $f(\vec{X})$ to be 1 at points $\vec{X} = (X,Y)$ in the slits and 0 everywhere else. Define $\vec{q} = \frac{k}{z}\vec{z}$, where $\vec{X} = (x, y)$, and $k = 2\pi/\lambda$. Let $I_0$ be the intensity at the origin of the observation screen.

Then, $I(\vec{q}) = I_0|G(\vec{q})|^2$, where $G(\vec{q}) = \int f(\vec{X})e^{-i\vec{q}\cdot\vec{X}} \, d^2X/A$, where $A$ is the total slit area.

Let us see how this comes about. Consider the displacement vector from a point in the slits, $(X,Y)$, to an observation point, $(x, y)$. Let $R = \sqrt{x^2 + y^2 + z^2}$ be the distance from the origin to the point on the observation screen. Then,

$$r = R[1 - \frac{2}{R^2}(xX + yY) + \frac{1}{R^2}(x^2 + y^2)]^{1/2} \approx R - \frac{x}{R}X - \frac{y}{R}Y.$$ \hfill (10.7.1)

Furthermore, $z >> x, y$ (small angle) implies $R \approx z$. I will only replace $R$ with $z$ when it appears anywhere except in the exponential. A little bit of surface area $d^2X$ will contribute a spherical wave $dE = (\frac{E}{z})e^{i(kR-\omega t)} \, d^2X$. Thus,

$$dE \approx \left(\frac{Ee^{i(kR-\omega t)}}{z}\right)e^{-i(k/R)(xX+yY)} \, d^2X = \left(\frac{Ee^{i(kR-\omega t)}}{z}\right)e^{-i\vec{q}\cdot\vec{X}} \, d^2X.$$ \hfill (10.7.2)

The total electric field is an integral over the slits

$$E = \left(\frac{Ee^{i(kR-\omega t)}}{z}\right) \int e^{-i\vec{q}\cdot\vec{X}} \, d^2X = \left(\frac{E\mathcal{A}e^{i(kR-\omega t)}}{z}\right) G(\vec{q}).$$ \hfill (10.7.3)

$\left|\frac{E\mathcal{A}e^{i(kR-\omega t)}}{z}\right|^2$ is simply the intensity incident at $x = y = 0$, which is the center of the observation screen. This is what we called $I_0$. Hence, $I(\vec{q}) = I_0|G(\vec{q})|^2$. This derivation is
essentially the same as that in Hecht 10.2.4 pp.464.

Now, let’s compute the integral.

\[
G(q) = \frac{1}{a^2} \int_{-a/2}^{a/2} \int_{-a/2}^{a/2} dX \, dY \, e^{-i(2\pi/\lambda R)(xX+yY)}
\]

\[
= \frac{1}{a^2} \left( -\frac{\lambda R}{2\pi i x} \right) \left( -\frac{\lambda R}{2\pi i y} \right) e^{-i(2\pi/\lambda R)xX} \bigg|_{-a/2}^{a/2} e^{-i(2\pi/\lambda R)yY} \bigg|_{-a/2}^{a/2}
\]

\[
= -\frac{\lambda^2 R^2}{4\pi^2 a^2 xy} \left( e^{i(2\pi/\lambda R)(a/2)x} - e^{-i(2\pi/\lambda R)(a/2)x} \right) \left( e^{i(2\pi/\lambda R)(a/2)y} - e^{-i(2\pi/\lambda R)(a/2)y} \right)
\]

\[
= \left( \frac{\lambda R}{\pi ax} \right) \sin \left( \frac{\pi ax}{\lambda R} \right) \left( \frac{\lambda R}{\pi ay} \right) \sin \left( \frac{\pi ay}{\lambda R} \right)
\]

\[
= \sin c \sin c, \quad (10.7.4)
\]

where \( \alpha = \frac{\pi a}{\lambda R} \) and \( \beta = \frac{\pi a}{\lambda R} \). This is just the product of two single slit diffraction patterns, as is to be expected. Define \( \vartheta \) and \( \varphi \) such that \( \sin \vartheta = x/R \) and \( \sin \varphi = y/R \). Note that these are not the same as spherical angles. Then, \( \alpha = \frac{\pi a}{\lambda R} \sin \vartheta \) and \( \beta = \frac{\pi a}{\lambda R} \sin \varphi \). Hence,

\[
I(\vartheta, \varphi) = I_0 \sin c^2 \alpha \sin c^2 \beta \quad (10.7.5)
\]

(b) Now, we have

\[
G(q) = \frac{1}{10a^2} \left[ \int_{-5a/2}^{-3a/2} + \int_{a/2}^{7a/2} \right] dX \int_{a/2}^{a/2} dY \, e^{-i(2\pi/\lambda R)(xX+yY)}
\]

\[
= \frac{1}{10a^2} \left( -\frac{\lambda R}{2\pi i x} \right) \left( -\frac{\lambda R}{2\pi i y} \right) e^{-i(2\pi/\lambda R)xX} \bigg|_{-5a/2}^{7a/2} e^{-i(2\pi/\lambda R)yY} \bigg|_{a/2}^{a/2}
\]

\[
= \frac{1}{10a^2} \left[ e^{3i\alpha} - e^{5i\alpha} + e^{-7i\alpha} - e^{-i\alpha} \right] \sin c \beta. \quad (10.7.6)
\]

It turns out that the square amplitude of this can be simplified significantly:

\[
I(\vartheta, \varphi) = \frac{I_0}{100a^2} \left[ \sin^4(2\alpha) + \frac{1}{4} (\sin(4\alpha) - 2\sin(6\alpha))^2 \right] \sin c^2 \beta. \quad (10.7.7)
\]
Radiation

- Approximations: \( d << \lambda << r \), where \( d \) is the separation of the charges of the electric dipole, \( \lambda \) is the radiation wavelength, and \( r \) is the distance to the point where we are measuring the radiation. The first is a dipole approximation and the second describes the radiation zone.

- Radiated power for electric dipole radiation: \( \langle P \rangle = \mu_0 p_0^2 \omega^4 / 12\pi c \), where \( p_0 \) is the maximum dipole moment.

- General dipole radiation: radiated power (not averaged) is \( P = \mu_0 \ddot{\mathbf{p}}^2 / 6\pi c \).
  - For \( p(t) = p_0 \cos(\omega t) \) this reproduces the oscillating dipole case.
  - For a point charge, \( \mathbf{p}(t) = q \mathbf{d}(t) \) and so \( \ddot{\mathbf{p}} = q \mathbf{a} \). Then, we derive the Larmor formula: \( P = \mu_0 q^2 a^2 / 6\pi c \).

- Radiation zone electric and magnetic fields for general dipole radiation: \( \mathbf{B}(\mathbf{r}, t) = -\frac{\mu_0}{4\pi c} \left[ \dot{\mathbf{r}} \times \ddot{\mathbf{p}} \right]_{\text{ret}} \) and \( \mathbf{E} = -c \dot{\mathbf{r}} \times \mathbf{B} \), where the subscript means that we must evaluate at the retarded time \( t - r/c \).

- The Larmor formula holds for \( v << c \). The Lienard generalization is unlikely to be important, but here it is anyway: \( P_{\text{Lienard}} = P_{\text{Larmor}} \gamma^6 \left[ 1 - \left( \frac{v \times \mathbf{a}}{c} \right)^2 \right] \).

- Radiation force (Abraham-Lorentz formula): \( \mathbf{F}_{\text{rad}} = \mu_0 q^2 \dot{\mathbf{a}} / 6\pi c \).
11.1 Antenna Impedance

[Fall 2006 Classical (Morning), Problem 5; Griffiths 11.3] We consider a dipole antenna of dimension $d$ much smaller than the wavelength $\lambda$ of the emitted wave. This antenna can be modeled as two charges at the two extremities varying sinusoidally with time, $q(t) = \pm q \cos(\omega t)$, and the average power emitted over $4\pi$ can be computed relatively easily to be proportional to the average of the square of the second time derivative of the dipole moment,

$$\langle P \rangle = \frac{\mu_0 (qd)^2 \omega^4}{12\pi c}.$$  

(11.1.1)

(a) If we neglect the propagation time along the dipole wires, what is the intensity of the input current to the antenna?

(b) By comparing the average of the current squared and the emitted power, argue that the real part of the antenna impedance as seen from the input circuit (that is the dissipative part) is

$$R = \frac{2\pi}{3} \sqrt{\frac{\mu_0}{\epsilon_0}} \left( \frac{d}{\lambda} \right)^2.$$  

(11.1.2)

(c) Compute the numerical value for $d = 0.3\lambda$ (which, strictly speaking, is outside the domain of applicability of the above formulae). This explains that for TV stations around 100 MHz, a dipole of the order of 1m is approximately adapted to a coaxial cable of impedance 75 $\Omega$.

SOLUTION:

(a) $| \! | q \omega \sin(\omega t) | \! |$, where the absolute values are there since we’re asked for the intensity, or amplitude, of the current, which is positive by definition.

(b) $P = I^2 R$ and so $\langle P \rangle = \frac{1}{2} q^2 \omega^2 R$ (from part (a)). Equating this to (11.1.1) yields

$$R = \frac{\mu_0 d^2 \omega^2}{6\pi c} = \frac{2\pi}{3} \sqrt{\frac{\mu_0}{\epsilon_0}} \left( \frac{d}{\lambda} \right)^2.$$  

(11.1.3)

where we used $\omega/c = 2\pi/\lambda$ and $\mu_0 c = \sqrt{\mu_0/\epsilon_0}$.

(c) The vacuum impedance is approximately $Z_0 = \sqrt{\mu_0/\epsilon_0} \approx 377 \Omega$. Thus,

$$R = (790 \Omega)(d/\lambda)^2 \approx 71 \Omega.$$  

(11.1.4)
11.2 Rotating Electric Dipole

[Griffiths 11.4] A rotating electric dipole can be thought of as the superposition of two oscillating dipoles, one along the x-axis, and the other along the y-axis, with the latter out of phase by 90°:

\[ \mathbf{p} = p_0[\cos(\omega t) \hat{x} + \sin(\omega t) \hat{y}] . \quad (11.2.1) \]

Using the principle of superposition, find the fields of the rotating dipole. Also find the Poynting vector and the intensity of the radiation. Sketch the intensity profile as a function of the polar angle \( \theta \), and calculate the total power radiated. Does the answer seem reasonable? (Note that power, being quadratic in the fields, does not satisfy the superposition principle. In this instance, however, it seems to. Can you account for this?)

**SOLUTION:**  (Thanks to Nesty)

I will prefer to use pretty much the only thing I remember for electric dipole radiation, which is that the radiation magnetic field is given by

\[ \mathbf{B}(\mathbf{r}, t) = -\frac{\mu_0}{4\pi r c} [\mathbf{r} \times \mathbf{p}]_{ret} , \quad (11.2.2) \]

In this case,

\[ \mathbf{p} = -p_0 \omega^2 [\cos(\omega t) \hat{x} + \sin(\omega t) \hat{y}] , \quad (11.2.3) \]

so that, after using \( \mathbf{r} \times \hat{x} = \cos \theta \hat{y} - \sin \theta \sin \phi \hat{z} \) and \( \mathbf{r} \times \hat{y} = -\cos \theta \hat{x} + \sin \theta \cos \phi \hat{z} \),

\[ \mathbf{B}(\mathbf{r}, t) = \frac{\mu_0 p_0 \omega^2}{4\pi r c} \left\{ \cos \theta \left[ -\sin(\omega t_r) \hat{x} + \cos(\omega t_r) \hat{y} \right] + \sin \theta \sin(\omega t_r - \phi) \hat{z} \right\} , \quad (11.2.4) \]

where \( t_r = t - r/c \). The electric field is \( \mathbf{E} = -\mathbf{r} \times \mathbf{B} \).

Since \( \mathbf{E} \perp \mathbf{B} \), the intensity is \( I = |\mathbf{S}| = \frac{c}{\mu_0} B^2 \). We first find \( B^2 \):

\[
B^2 = \left( \frac{\mu_0 p_0 \omega^2}{4\pi r c} \right)^2 \left\{ \cos^2 \left[ \sin^2(\omega t_r) + \cos^2(\omega t_r) \right] + \sin^2 \theta \sin^2(\omega t_r - \phi) \right\} \\
= \left( \frac{\mu_0 p_0 \omega^2}{4\pi r c} \right)^2 \left\{ \cos^2 \theta + \sin^2 \theta \sin^2(\omega t_r - \phi) \right\} \\
= \left( \frac{\mu_0 p_0 \omega^2}{4\pi r c} \right)^2 \left[ 1 - \sin^2 \theta \cos^2(\omega t_r - \phi) \right] . \quad (11.2.5)
\]

We also know that the direction of \( \mathbf{S} \) is \( \hat{r} \). Thus,

\[ \mathbf{S} = \frac{\mu_0}{c} \left( \frac{p_0 \omega^2}{4\pi r c} \right)^2 \left[ 1 - \sin^2 \theta \cos^2(\omega t_r - \phi) \right] \hat{r} . \quad (11.2.6) \]

Time-averaging simply turns the \( \cos^2(\omega t_r - \phi) \) term into \( 1/2 \):

\[ \mathbf{S} = \frac{\mu_0}{c} \left( \frac{p_0 \omega^2}{4\pi} \right)^2 \left( 1 - \frac{1}{2} \sin^2 \theta \right) \hat{r} . \quad (11.2.7) \]

The angular power distribution is

\[ \frac{dP}{d\Omega} = r^2 \mathbf{S} \cdot \hat{r} = \frac{\mu_0}{c} \left( \frac{p_0 \omega^2}{4\pi} \right)^2 \left( 1 - \frac{1}{2} \sin^2 \theta \right) = \frac{\mu_0}{2c} \left( \frac{p_0 \omega^2}{4\pi} \right)^2 \left( 1 + \cos^2 \theta \right) . \quad (11.2.8) \]
Chapter 11. Radiation

Figure 11.1: Rotating dipole radiated power angular distribution.

The average power radiated is

$$\bar{P} = \frac{\mu_0}{2c} \left( \frac{p_0\omega^2}{4\pi} \right)^2 \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta (1 + \cos^2 \theta) = \frac{\mu_0 p_0^2 \omega^4}{6\pi c} \quad (11.2.9)$$

This is precisely twice as much as one dipole. So, it seems as though the power satisfies superposition. However, this is only a coincidence due to the fact that the two dipoles are mutually out of phase. That is, schematically, $\mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2$, separated out into the contributions from each dipole. But, $\mathbf{B}_1$ contains a $\cos(\omega t_r)$ factor, whereas $\mathbf{B}_2$ contains $\sin(\omega t_r)$. The Poynting vector is proportional to the square of $\mathbf{B}$. Superposition fails because there is a cross term, $2\mathbf{B}_1 \cdot \mathbf{B}_2$. However, this term vanishes after time-averaging since the average of $\cos(\omega t_r) \sin(\omega t_r)$ is zero.
11.3 Oscillating Charge Radiation

[Griffiths 11.21] A particle of mass \(m\) and charge \(q\) is attached to a spring with force constant \(k\), hanging from the ceiling. Its equilibrium position is a distance \(h\) above the floor. It is pulled down a distance \(d\) below equilibrium and released, at time \(t = 0\).

(a) Under the usual assumptions \((d << \lambda << h)\), calculate the intensity of the radiation hitting the floor, as a function of the distance \(R\) from the point directly below \(q\). [Note: The intensity here is the average power per unit area of the floor.] At what \(R\) is the radiation most intense? Neglect the radiative damping of the oscillator.

(b) As a check on your formula, assume the floor is of infinite extent, and calculate the average energy per unit time striking the entire floor. Is it what you would expect?

(c) Because it is losing energy in the form of radiation, the amplitude of the oscillation will gradually decrease. After what time \(\tau\) has the amplitude been reduced to \(d/e\)? (Assume the fraction of the total energy lost in one cycle is very small.)

SOLUTION:

(a) This is like an oscillating electric dipole \(p_0 = qd\) and frequency \(\omega = \sqrt{k/m}\). The averaged Poynting vector is

\[
\langle S \rangle = \left( \frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c} \right) \frac{\sin^2 \theta}{r^2} \hat{r}.
\]  

(11.3.1)

Let \(\hat{z}\) be the direction normal to the floor and pointing downwards. Then, the intensity hitting the floor is

\[
I = \langle S \rangle \cdot \hat{z} = \left( \frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c} \right) \frac{\sin^2 \theta \cos \theta}{r^2}.
\]  

(11.3.2)

After substituting in \(R = r \sin \theta, h = r \cos \theta\) and \(r^2 = R^2 + h^2\), we get

\[
I = \left( \frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c} \right) \frac{R^2 h}{(R^2 + h^2)^{3/2}}.
\]  

(11.3.3)

Finding the value of \(R\) that maximizes this is straightforward:

\[
\text{max intensity at } R = \sqrt{2/3}h.
\]  

(11.3.4)

Suppose you wanted to start from \(\mathbf{B}(r, t) = -\frac{\mu_0}{4\pi c} \left[ \hat{r} \times \mathbf{p} \right]_{\text{ret}}\). I would set the origin at the equilibrium position of the mass with the \(z\)-axis pointing vertically downward so that the position of the mass is \(z(t) = d \cos(\omega t)\) and it is essentially a dipole at the origin with dipole moment \(\mathbf{p}(t) = qz(t) \hat{z}\).

(b) The total power incident on the floor is

\[
P = 2\pi \left( \frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c} \right) h \int_{0}^{\infty} \frac{R^3 dR}{(R^2 + h^2)^{5/2}}.
\]  

(11.3.5)

Denote the integral by \(I\). First, set \(x = R^2\). One integration by parts yields

\[
I = \frac{1}{2} \int_{0}^{\infty} \frac{x \, dx}{(x + h^2)^{5/2}} = -\frac{x}{3(x + h^2)^{3/2}} \bigg|_{0}^{\infty} + \frac{1}{3} \int_{0}^{\infty} \frac{dx}{(x + h^2)^{3/2}}
\]

\[
= -\frac{2}{3\sqrt{x + h^2}} \bigg|_{0}^{\infty} = 2/3h.
\]  

(11.3.6)
Therefore, we have the power:

\[ P = \frac{\mu_0 q^2 d^2 \omega^4}{24\pi c} \]  

(11.3.7)

This is half of the total radiated power, as expected.

(c) Since our \( P \) is only half of the total radiated power, \( dU/dt = -2P \), where \( U = \frac{1}{2}kx^2 \), is the spring energy (note: the gravitational energy at equilibrium obviously cannot be radiated away!) This yields the differential equation

\[ \frac{d}{dt}(d^2) = -\frac{\mu_0 q^2 \omega^4}{6\pi kc} (d^2), \]

(11.3.8)

which is solved by a simple exponential decay. We find \( d = d_0 e^{-t/\tau} \) with

\[ \tau = \frac{12\pi kc}{\mu_0 q^2 \omega^4} = \frac{12\pi m^2 c}{\mu_0 q^2 k}, \]

(11.3.9)

where we used \( \omega = \sqrt{k/m} \).
11.4 Array of Seven Dipole Antennae

[Fall 2009 Classical (Morning), Problem 4] Seven antennae, radiating as electric dipoles polarized along the $\hat{z}$ direction, are placed along the $x$-axis in the $xy$-plane at $x = 0, \pm \lambda/2, \pm \lambda, \pm 3\lambda/2$. The antennae all radiate at wavelength $\lambda$ and are in phase.

(a) Calculate the angular distribution of the radiated power as a function of the polar and azimuthal angles, $\theta$ and $\phi$. Neglect any constant multiplying prefactors.

(b) Consider the direction in which the radiated intensity is maximum for this array and for a single dipole antenna. How do these intensities compare?

SOLUTION:

(a) Let $\xi = \lambda/2$ be the distance between adjacent antennae. The displacement vector that takes us from one the source at $x = j\xi$ (for $j = -3, \ldots, 3$) to the field point, $r$, is $\mathbf{z}_j = r\hat{r} - j\xi \hat{x}$. The length of this vector is

$$\mathbf{z}_j \approx \sqrt{r^2 - 2j\xi r \hat{r} \cdot \hat{x}} \approx r \left[1 - (j\xi/r) \sin \theta \cos \phi\right]. \quad (11.4.1)$$

Set each dipole to be $p(t) = p(t) \hat{z}$ where $p(t) = p_0 e^{i\omega t}$. Then, $\mathbf{p} = -\omega^2 p_0 e^{i\omega t} \hat{z}$. Thus, the magnetic field produced by the source at $x = j\xi$ is

$$\mathbf{B}_j(r, t) = -\frac{\mu_0}{4\pi} \frac{\mathbf{p}_0}{r} \left[\hat{r} \times \mathbf{p}\right]_{\text{ret}}. \quad (11.4.2)$$

Since there is already a $1/r$ in the $1/\mathbf{z}_j$ term, we need to keep the terms in the cross-product that are $r$-independent. This means dropping all terms involving $\xi$ except in the exponential:

$$\mathbf{B}_j(r, t) = \frac{\mu_0 \mathbf{p}_0 \omega^2}{4\pi r c} e^{i\omega(t-\mathbf{z}_j/c)\hat{r} \times \hat{z}}. \quad (11.4.3)$$

We can write $\hat{r}$ in cylindrical coordinates as $\hat{r} = \sin \theta \hat{\rho} + \cos \theta \hat{z}$. Then, using $\hat{\rho} \times \hat{z} = -\hat{\phi}$ gives

$$\mathbf{B}_j(r, t) = -\frac{\mu_0 \mathbf{p}_0 \omega^2}{4\pi r c} e^{i\omega(t-r/c) \sin \theta \hat{\phi} e^{i\omega(t-\mathbf{z}_j/c) \sin \theta \cos \phi}}. \quad (11.4.4)$$

Note that $\omega \lambda/2\pi c = 1$ and $\xi = \lambda/2$. Thus, $\omega \xi/c = \pi$. Set $\delta \equiv (\pi/2) \sin \theta \cos \phi$, so that the total magnetic field is

$$\mathbf{B}(r, t) = -\frac{\mu_0 \mathbf{p}_0 \omega^2}{4\pi r c} e^{i\omega(t-r/c) \sin \theta \hat{\phi}} \sum_{j=-3}^{3} e^{2ij\delta}. \quad (11.4.5)$$

Of course, the actual field is the real part of this alone. We need not compute the electric field to find the Poynting vector. We simply have $\mathbf{S} = \frac{\mu_0}{\varepsilon_0} (\mathbf{B} \cdot \mathbf{B}) \hat{r}$, where the field here is understood to be just the real part of (11.4.5). Furthermore, the sum of the exponentials is simply

$$\sum_{j=-3}^{3} e^{2ij\delta} = 1 + 2 \cos 2\delta + 2 \cos 4\delta + 2 \cos 6\delta \equiv f(\delta). \quad (11.4.6)$$

Lastly, the Poynting vector contains a factor $\cos^2[\omega(t - r/c)]$ whose time average is $1/2$. Hence,

$$\mathbf{S} = \frac{\mu_0}{2c} \left(\frac{\mathbf{p}_0 \omega^2}{4\pi r c}\right)^2 \sin^2 \theta f(\delta)^2 \hat{r}. \quad (11.4.7)$$
It turns out that $f(\delta)$ has a neat form, so that the average angular power distribution is found below and plotted in Figure 11.2.

$$\frac{dP}{d\Omega} = r^2 \mathbf{\hat{S}} \cdot \mathbf{\hat{r}} = \frac{\mu_0}{2c} \left( \frac{p_0 \omega^2}{4\pi} \right)^2 \sin^2 \theta f(\delta)^2 \propto \sin^2 \theta \frac{\sin^2(7\delta)}{\sin^2(\delta)}. \tag{11.4.8}$$

Figure 11.2: Fall 2009 Classical (Morning), Problem 4

(b) The terms with cosines of sines and cosines are all maximized when all of their arguments vanish (i.e. when all the cosines are +1). This requires either $\cos \phi = 0$ or $\sin \theta = 0$. However, the latter will make the whole thing vanish since there is also a $\sin^2 \theta$ factor outside. Thus, we need $\phi = 0$ or $\phi = \pi$, and after that, up to numerical factors, we’re back to the just the pure $\sin^2 \theta$ behavior, which is the same for the single dipole case and hence the direction of maximum radiation is the same. Therefore, the difference is: for a single dipole, the directions for maximum radiation are $\theta = \pi/2$ and $\phi$ is arbitrary, whereas for the array we only have maxima when $\theta = \pi/2$ and $\phi = 0, \pi$. This should be clear from the diagram.
Chapter 12

Relativity

- Minkowski metric: $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$.
- Contravariant vector: $x^\mu = (t, \mathbf{x})$. Covariant vector: $x_\mu = g_{\mu\nu}x^\nu = (-t, \mathbf{x})$.
- Current density vector: $J^\mu = (\rho, \mathbf{J})$.
- Contravariant vector potential: $A^\mu = (\phi, \mathbf{A})$.
- Field strength: $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ where $\partial^\mu = \partial/\partial x^\mu$. The explicit form is in Griffiths 12.118, p.536.
- Lorentz transformation of vector: $x'^\mu = \Lambda^\mu_\nu x^\nu$. You can replace $x^\mu$ with $A^\mu$ or $J^\mu$ or any other vector.
- Lorent transformation of second-rank tensor: $F'^{\mu\nu} = \Lambda^\mu_\rho \Lambda^\nu_\sigma F^{\rho\sigma}$.
- Energy-momentum tensor: $T^{\mu\nu} = -\frac{1}{\mu_0} \left[ F^{\mu\alpha} F_\alpha^{\nu} + \frac{1}{4} \eta^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right]$. This can also be written as $T^{\mu\nu} = -\frac{1}{\mu_0} \left[ \eta^{\mu\alpha} \eta^{\beta\gamma} \delta^\delta \delta^{\nu} + \frac{1}{4} \eta^{\mu\nu} \eta^{\sigma\gamma} \eta^{\beta\delta} \right] F_{\alpha\beta} F^{\gamma\delta}$. 

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12.1 Point Charge Field Transformation

[Fall 2005 Classical (Afternoon), Problem 6] A point charge $q$ is at rest at the origin in system $S_0$. What are the electric and magnetic fields of this same charge in system $S_1$, which moves to the right at speed $v_0$ relative to $S_0$?

**SOLUTION:**

In $S_0$, there is no magnetic field and the electric field is

$$\mathbf{E} = \frac{q\mathbf{\hat{r}}}{4\pi\varepsilon_0 r^2} = \frac{q}{4\pi\varepsilon_0 r^3} (x\mathbf{\hat{x}} + y\mathbf{\hat{y}} + z\mathbf{\hat{z}}). \quad (12.1.1)$$

Therefore, the field strength is

$$F^{\mu\nu} = \frac{q}{4\pi\varepsilon_0 r^3} \begin{pmatrix} 0 & x & y & z \\ - & 0 & 0 & 0 \\ - & 0 & 0 & 0 \\ - & 0 & 0 & 0 \end{pmatrix}, \quad (12.1.2)$$

where the $-$'s signs just signify negative of the first row entries.

Let us boost in the $x$ direction by $\beta = v_0$ (note that $c = 1$):

$$\Lambda^\nu_\mu = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (12.1.3)$$

The transformed field strength is

$$F^{\mu'\nu'} = \Lambda^\nu_\mu F^{\mu\nu} = (\Lambda F \Lambda^T)^{\mu'\nu'} \quad (12.1.4)$$

The matrix multiplication yields

$$F^{\mu'\nu'} = \frac{q}{4\pi\varepsilon_0 r^3} \begin{pmatrix} 0 & \gamma y & \gamma z \\ - & 0 & -\beta\gamma y & -\beta\gamma z \\ - & 0 & 0 & 0 \\ - & 0 & 0 & 0 \end{pmatrix}. \quad (12.1.5)$$

The unit vectors are unchanged by a boost, thankfully. However, if we want to express this in terms of the primed coordinates, we must replace $x$ with $\gamma(\beta x' + ct')$. The other spatial coordinates are unchanged ($y' = y$ and $z' = z$). Thus,

$$\mathbf{E}' = \frac{\gamma q}{4\pi\varepsilon_0} \left[ \frac{(\beta x' + ct')\mathbf{\hat{x}} + y'\mathbf{\hat{y}} + z'\mathbf{\hat{z}}}{\gamma^2(\beta x' + ct')^2 + y'^2 + z'^2} \right],$$

$$\mathbf{B}' = -\frac{\beta\gamma q}{4\pi\varepsilon_0} \left[ \frac{y'\mathbf{\hat{y}} + z'\mathbf{\hat{z}}}{\gamma^2(\beta x' + ct')^2 + y'^2 + z'^2} \right]. \quad (12.1.6)$$
12.2 Moving Dielectric Interface

[Kevin G.] Electromagnetic radiation polarized in the plane of incidence is incident upon a planar interface between two regions with dielectric constants \( \varepsilon_1 \) and \( \varepsilon_2 \) (it is incident from 1 to 2.) In the lab frame the dielectrics are moving with speed \( \beta \) away from the radiation source in the direction normal to their planar interface. The angle of incidence relative to the normal in the lab frame is \( \theta_i \). Express the reflection and transmission angles in the lab frame, \( \theta_R \) and \( \theta_T \), in terms of \( \theta_i \), and calculate the effect of the motion on the reflection and transmission coefficients.

**SOLUTION:**

The entire derivation in Griffiths pp.386-392 holds in the rest frame of the dielectrics. I’m not going to rederive all of that here. Just review it on your own. The picture relevant to the problem is Figure 9.15 Griffiths p.389.

We must relate the fields in the rest frame of the dielectrics to those in the lab frame. To go from the dielectric rest frame to the lab frame, we must boost in the \(-\hat{z}\) direction by \( \beta \). The appropriate boost matrix thus has positive off-diagonal entries as below. This once, I will write down every step in the transformation of the field tensor: \( F^{\mu\nu} = \Lambda^\mu_\mu \Lambda^\nu_\nu F'^{\mu\nu} = \Lambda^\mu_\mu F'^{\mu\nu} (\Lambda^T)^\nu_\nu = (\Lambda F' \Lambda^T)^{\mu\nu} \). To reduce clutter, I will set the speed of light in a general dielectric equal to 1. At the end, we will reintroduce it via \( \mathbf{E} \rightarrow \mathbf{E}/v \). Also, we will refrain from putting primes everywhere until the very end.

\[
F A^T = \begin{pmatrix}
0 & E_x & E_y & E_z \\
-E_x & 0 & B_z & -B_y \\
-E_y & -B_z & 0 & B_z \\
-E_z & B_y & -B_x & 0
\end{pmatrix}
\begin{pmatrix}
\gamma & 0 & 0 & \beta \gamma \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\beta \gamma & 0 & 0 & \gamma
\end{pmatrix}
\begin{pmatrix}
\beta \gamma E_x \\
\gamma E_y \\
\gamma E_z \\
0
\end{pmatrix}
\]

(12.2.1)

Then, multiplying by \( \Lambda \) gives

\[
\Lambda F A^T = \begin{pmatrix}
\gamma & 0 & 0 & \beta \gamma \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\beta \gamma & 0 & 0 & \gamma
\end{pmatrix}
\begin{pmatrix}
\beta \gamma E_x & E_x & E_y & \gamma E_z \\
\beta \gamma E_z & 0 & B_z & -\gamma (B_y + \beta E_x) \\
\gamma (E_y - \beta B_z) & -B_z & 0 & \gamma (B_x - \beta E_y) \\
-\gamma E_x & B_y & -B_z & -\beta \gamma E_z
\end{pmatrix}
\begin{pmatrix}
\gamma E_x + \beta B_y \\
\gamma E_y - \beta B_x \\
\gamma E_z \\
0
\end{pmatrix}
\]

(12.2.2)

Now, reintroducing \( v \) and writing \( F' \) instead of \( F \) above (i.e. \( \Lambda F' \Lambda^T \)) gives

\[
E_x = \gamma (E'_x + \beta B'_y), \quad B_x = \gamma (B'_x - \beta E'_y), \\
E_y = \gamma (E'_y - \beta B'_x), \quad B_y = \gamma (B'_y + \beta E'_x), \\
E_z = E'_z, \quad B_z = B'_z.
\]

(12.2.3)

This relates the fields in the lab frame (no primes) to the fields in the dielectric frame (with primes). The next quantities we must relate are the wavevectors in both frames. This is done by \( k'^\mu = \Lambda^\mu_\mu k'^\mu \). Remember that \( k^0 = \omega/v = k \) where \( k^2 = k_x^2 + k_y^2 + k_z^2 \). You can work out for
Write the incident electric field in the rest frame as
\[ k = \gamma (k' + \beta k_z'), \quad k_x = k'_x, \]
\[ k_z = \gamma (k'_z + \beta k'), \quad k_y = k'_y. \]  
(12.2.4)

In the rest frame, \( \theta'_l = \theta'_r \) and \( n_1 \sin \theta'_l = n_2 \sin \theta'_r \). Let the incident wavevector in the lab frame be \( k_I(1, \sin \theta_I, 0 \cos \theta_I) \), then the incident wavevector in the rest frame is given via (12.2.4) with \( \beta \to -\beta \) since we are boosting from the lab to the rest frame, which is a positive-\( z \) boost:
\[ k'_I(1, \sin \theta'_I, 0, \cos \theta'_I) = k_I \{ \gamma (1 - \beta \cos \theta_I), \sin \theta_I, 0, \gamma (\cos \theta_I - \beta) \}. \]  
(12.2.5)

The reflected wavevector in the rest is
\[ k'_R = k_I \{ \gamma (1 - \beta \cos \theta_I), \sin \theta_I, 0, -\gamma (\cos \theta_I - \beta) \}. \]  
(12.2.6)

Boosting back gives us the reflected wavevector in the lab frame
\[ k_R = k_I \{ \gamma \left[ \gamma (1 - \beta \cos \theta_I) - \beta \gamma (\cos \theta_I - \beta) \right], \sin \theta_I, 0, \gamma \left[ -\gamma (\cos \theta_I - \beta) + \beta \gamma (1 - \beta \cos \theta_I) \right] \} \]
\[ = k_I \left\{ \gamma^2 (1 - 2 \beta \cos \theta_I + \beta^2), \sin \theta_I, 0, -\gamma^2 \left[ (1 + \beta^2) \cos \theta_I - 2 \beta \right] \right\}. \]  
(12.2.7)

Therefore, the reflection angle in the lab frame is given by
\[ \tan \theta_R = \frac{\sin \theta_I}{\gamma^2 \left[ (1 + \beta^2) \cos \theta_I - 2 \beta \right]} = \frac{(1 - \beta^2) \sin \theta_I}{(1 + \beta^2) \cos \theta_I - 2 \beta}. \]  
(12.2.8)

From (12.2.5) we have
\[ k'_I = k_I \gamma (1 - \beta \cos \theta_I), \quad \sin \theta'_I = \frac{\sin \theta_I}{\gamma (1 - \beta \cos \theta_I)}. \]  
(12.2.9)

The transmitted wavevector in the rest frame is (c.f. Griffiths p.388)
\[ k'_T = \frac{n_2}{n_1} k'_I \left( 1, \frac{n_1}{n_2} \sin \theta'_I, 0, \sqrt{1 - \left( \frac{n_1}{n_2} \right)^2 \sin^2 \theta'_I} \right). \]  
(12.2.10)

Combining the last two equations and boosting back to the lab frame yields
\[ \tan \theta_T = \frac{n_1 \sin \theta_I}{n_2 \left( 1 - \beta \cos \theta_I \right)} \left[ \left( 1 - \frac{n_1^2 (1 - \beta^2) \sin^2 \theta_I}{n_2^2 (1 - \beta \cos \theta_I)^2} \right)^{1/2} + \beta \right]^{-1}. \]  
(12.2.11)

**Aside:** I'll leave it up to you to prove the useful identity
\[ \gamma^2 (1 - \beta \cos \theta_I)^2 = \sin^2 \theta_I + \gamma^2 (\cos \theta_I - \beta)^2. \]  
(12.2.12)

Using this, we may neatly write \( \cos \theta'_I \) as
\[ \cos \theta'_I = \frac{\cos \theta_I - \beta}{1 - \beta \cos \theta_I}, \]  
(12.2.13)

which is useful if you start with the wavevectors in the rest frame first and relate all of the lab frame wavevectors to these.

Write the incident electric field in the rest frame as
\[ \mathbf{E}'_I(x', t') = \mathbf{E}_I e^{i(k'_I x' - \omega' t')}, \]  
(12.2.14)
with similar expressions for the reflected and transmitted waves, just replacing $I$ with $R$ or $T$, and where $E'_I$ is some constant vector.

Let $E'_I = |E'_I|$. In the rest frame, the incident field amplitudes are given by

$$
\begin{align*}
E'_{Ix} &= E'_I \cos \theta'_I, & B'_{Iz} &= 0, \\
E'_{Iy} &= 0, & B'_{Iy} &= E'_I/v_1, \\
E'_{Iz} &= -E'_I \sin \theta'_I, & B'_{Iz} &= 0.
\end{align*}
$$

(12.2.15)

Using (12.2.3), we may write the field amplitudes in the lab frame:

$$
\begin{align*}
E'_{Ix} &= \gamma (\cos \theta'_I + \beta) E'_I, & B'_{Iz} &= 0, \\
E'_{Iy} &= 0, & B'_{Iy} &= \gamma (1 + \beta \cos \theta'_I) E'_I/v_1, \\
E'_{Iz} &= -E'_I \sin \theta'_I, & B'_{Iz} &= 0.
\end{align*}
$$

(12.2.16)

All we actually need from this is

$$
E_I = E'_I \gamma (1 + \beta \cos \theta'_I).
$$

(12.2.17)

Doing the same for the reflected waves yields

$$
E_R = E'_R \gamma (1 - \beta \cos \theta'_I).
$$

(12.2.18)

Hence, after using (12.2.14), the reflection coefficient is

$$
R = \left( \frac{E_R}{E_I} \right)^2 = \frac{1 - \beta \cos \theta'_I}{1 + \beta \cos \theta'_I} \left( \frac{E'_R}{E'_I} \right)^2 = \left( \frac{(1 - 2\beta \cos \theta'_I + \beta^2)}{1 - \beta^2} \right) R',
$$

(12.2.19)

where $R'$ is the reflection coefficient in the rest frame, and $T = 1 - R$. 

12.3 Moving Charge by a Current

(Spring 2005 Classical (Morning), Problem 6) A very thin, long, electrically neutral wire is fixed in the laboratory frame and carries a uniform current, \( I \), in the \(-\hat{x}\) direction. An electron moves outside the wire with an (instantaneous) velocity \( \mathbf{v} = v\hat{x} \), with \( v \ll c \). The (instantaneous) distance from the electron to the wire is \( y \). It just so happens that \( v \) is also equal to the magnitude of the drift velocity of the charge carriers in the wire, \( v_d \).

(a) In the lab frame (in which the wire is at rest and the electron in question moves to the right), calculate the force on the electron due to the current-carrying wire.

(b) In the frame of the electron (in which the electron is at rest and the wire bodily moves to the left), calculate the force on the electron. (Do not neglect relativistic corrections.)

(c) Compare your answers to parts (a) and (b), and comment on the nature of the force in each case. What qualitative differences (in force magnitude and force “origin”) would you expect if the force on the electron were measured in a reference frame moving to the right at velocity \( v' = v/2 \) with respect to the fixed lab frame?

SOLUTION:

(a) In the lab frame, the wire only produces a magnetic field, which we can compute using Ampère’s law: \( 2\pi r B = \oint \mathbf{B} \cdot d\ell = \mu_0 I_{\text{enc}} = \mu_0 I \). Hence, \( B = \mu_0 I / 2\pi r \).

Define cylindrical coordinates with \( x \) taking the usual place of \( z \), \( y \) that of \( x \) and \( z \) that of \( y \). Thus, the polar angle, \( \theta \), is with respect to \( \hat{x} \), the azimuthal angle, \( \phi \), is with respect to \( \hat{y} \) going counter clockwise in the \( y - z \) plane and \( \rho \) is the radial distance from the origin in the \( y - z \) plane. Then, \( B = -\frac{\mu_0 I}{2\pi \rho} \hat{\phi} \).

At the instantaneous position of the electron, \( B = -\frac{\mu_0 I}{2\pi y} \hat{y} \). The Lorentz force is

\[
\mathbf{F} = q\mathbf{v} \times \mathbf{B} = -e(v\hat{x}) \times (-\frac{\mu_0 I}{2\pi y} \hat{y}) = -\frac{\mu_0 I e v}{2\pi y} \hat{y}.
\] (12.3.1)

(b) Write \( \dot{\phi} = -\sin \phi \hat{y} + \cos \phi \hat{z} \). Thus, in the lab frame, the electromagnetic field is

\[
\begin{align*}
E_x &= 0, & B_x &= 0, \\
E_y &= 0, & B_y &= B \sin \phi, \\
E_z &= 0, & B_z &= -B \cos \phi,
\end{align*}
\] (12.3.2)

where \( B = \mu_0 I / 2\pi \rho \).

We need to know how these transform under a boost with speed \( v \) in the \(+\hat{x}\) direction.

Method 1: Set \( c = 1 \). Write the field strength tensor:

\[
F^{\mu\nu} = \begin{pmatrix}
0 & E_x & E_y & E_z \\
-E_x & 0 & B_z & -B_y \\
-E_y & -B_z & 0 & 0 \\
-E_z & B_y & 0 & 0
\end{pmatrix} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & B_z & -B_y \\
0 & -B_z & 0 & 0 \\
0 & B_y & 0 & 0
\end{pmatrix}.
\] (12.3.3)

Recall that \( F' = F^{\mu'\nu'} = \Lambda^{\mu'}_{\mu} \Lambda^{\nu'}_{\nu} F^{\mu\nu} = (AF\Lambda^T)^{\mu'\nu'} \), where \( F' \) is the field strength tensor in the boosted frame. We could simply multiply these matrices out, which is not difficult since
most of the entries are zero. We could write

\[ F^{\mu \nu} = B_z (\delta^\mu_1 \delta^\nu_2 - \delta^\mu_2 \delta^\nu_1) - B_y (\delta^\mu_1 \delta^\nu_3 - \delta^\mu_3 \delta^\nu_1) \]

\[ \Lambda^\nu_\mu = \gamma (\delta^\nu_0 \delta^\mu_0 + \delta^\nu_1 \delta^\mu_1 - \beta \gamma (\delta^\nu_0 \delta^\mu_1 + \delta^\nu_1 \delta^\mu_0) + \delta^\nu_2 \delta^\mu_2 + \delta^\nu_3 \delta^\mu_3. \] (12.3.4a)

Thus, we have

\[ \Lambda^{\nu_1}_1 = \gamma (\delta^{\nu_1}_1 - \beta \delta^{\nu_0}_0), \quad \Lambda^{\nu_2}_2 = \delta^{\nu_2}_2, \quad \Lambda^{\nu_3}_3 = \delta^{\nu_3}_3. \] (12.3.5)

Therefore, the field strength tensor in the boosted frame is

\[ F^{\mu \nu'} = \Lambda^{\mu \nu'} \Lambda^\nu_\mu F^{\mu \nu} \]

\[ = B_z (\Lambda^{\mu}_1 \Lambda^\nu_2 - \Lambda^{\mu}_2 \Lambda^\nu_1) - B_y (\Lambda^{\mu}_1 \Lambda^\nu_3 - \Lambda^{\mu}_3 \Lambda^\nu_1) \]

\[ = \gamma B_z \left[ (\delta^{\mu_1}_1 - \beta \delta^{\mu_0}_0) \delta^{\nu_2}_2 - (\delta^{\mu_2}_2 - \beta \delta^{\mu_0}_0) \delta^{\nu_1}_1 \right] - \gamma B_y \left[ (\delta^{\mu_1}_1 - \beta \delta^{\mu_0}_0) \delta^{\nu_3}_3 - (\delta^{\nu_1}_1 - \beta \delta^{\nu_0}_0) \delta^{\nu_3}_3 \right] \]

\[ = -\beta \gamma B_z (\delta^{\mu_0}_0 \delta^{\nu_2}_2 - \delta^{\mu_2}_2 \delta^{\nu_0}_0) + \gamma B_z (\delta^{\mu_1}_1 \delta^{\nu_2}_2 - \delta^{\nu_1}_1 \delta^{\mu_0}_0) \]

\[ + \beta \gamma B_y (\delta^{\mu_0}_0 \delta^{\nu_3}_3 - \delta^{\nu_0}_0 \delta^{\nu_3}_3) - \gamma B_y (\delta^{\nu_1}_1 \delta^{\mu_3}_3 - \delta^{\mu_1}_1 \delta^{\nu_3}_3). \] (12.3.6)

In matrix form, this reads

\[ F^{\mu \nu'} = \begin{pmatrix} 0 & 0 & -\beta \gamma B_z & \beta \gamma B_y \\ 0 & 0 & \beta \gamma B_z & -\gamma B_y \\ -\beta \gamma B_y & -\gamma B_z & 0 & 0 \\ -\beta \gamma B_y & 0 & \gamma B_z & 0 \end{pmatrix}. \] (12.3.7)

In other words,

\[ E'_x = 0, \quad B'_x = 0, \]

\[ E'_y = \beta \gamma B \cos \phi, \quad B'_y = \gamma B \sin \phi, \] (12.3.8)

\[ E'_z = \beta \gamma B \sin \phi, \quad B'_z = -\gamma B \cos \phi, \]

or, more succinctly,

\[ \mathbf{E}' = \beta \gamma B \hat{\rho}, \quad \mathbf{B}' = -\gamma B \hat{\phi}. \] (12.3.9)

So far, this has all been exact. If \( v << c \), then \( \beta << 1 \) and \( \gamma = (1 - \beta^2)^{-1/2} \approx 1 + \mathcal{O}(\beta^2) \). Hence,

\[ \mathbf{E}' \approx \beta B \hat{\rho}, \quad \mathbf{B}' \approx -\gamma B \hat{\phi}. \] (12.3.10)

If this boosts to the rest frame of the electron then \( \beta = \frac{v}{c} \). Since the electron is at rest it only feels the electric field. The force is \( \mathbf{F}' = q \mathbf{E}' = -e \beta \gamma B \hat{\rho} \approx -\frac{mev}{2\pi \gamma} \hat{y} \). Putting \( c \) back in, we need to multiply \( \mathbf{E}' \) by \( c \) so that electric field has units of \( c \times \) magnetic field. This gives

\[ \mathbf{F}' = -\frac{mev}{2\pi \gamma} \hat{y}, \] (12.3.11)

which is the same as in the rest frame!

**Method 2:** In the lab frame, let \( \lambda_+ = \lambda_0 \) and \( \lambda_- = -\lambda_0 \) be the positive and negative charge densities, which are equal in magnitude since the wire is electrically neutral in this frame.
In this frame, the positive charges are not moving and the negative charges are moving at \( v \hat{x} \).

There are two effects of boosting to the electron rest frame: (1) since the electrons in the wire are moving with the same velocity, they are also now at rest; by reverse length contraction, they are spread further apart and \( \lambda'_- < \lambda_- \); (2) now the positive charges are moving with velocity \(-v\hat{x}\) and, by length contraction, \( \lambda'_+ > \lambda_+ \). These two things imply that the wire is no longer neutral in this frame, and thus an electric field is produced in addition to a magnetic field due to the current carried by the positive charges.

Quantitatively, \( \lambda'_- = \lambda_- / \gamma = -\lambda_0 / \gamma \) and \( \lambda'_+ = \gamma \lambda_+ = \gamma \lambda_0 \). Thus, the current, which is now carried only by the positive charges, is \( I' = \beta \lambda'_+ = \gamma \beta \lambda_0 \). We can identify \( \beta \lambda_0 \) as the current in the lab frame, \( I \), since in that frame, \( \lambda_0 \) is the magnitude of the negative charge density and \( \beta \) is the speed of those negative charges. Hence, \( I' = \gamma I \) and so \( B' = \gamma B \). The total charge density is \( \lambda' = \lambda'_+ + \lambda'_- = \frac{\lambda_0}{(\gamma^2 - 1)} = \gamma \beta^2 \lambda_0 = \beta \gamma I \). Hence, in magnitude, \( E' = \beta \gamma B \). This agrees with Method 1.

(c) For \( v << c \), where we can neglect the \( \gamma \)'s, the forces are equal, but in the lab frame the force is purely magnetic, whereas in the electron rest frame, the force is purely electric. In an in-between frame, the force would be the same, but it would have both electric and magnetic contributions.
Chapter 13

Optics

- Let SLC denote “the side of the lens/mirror from which the light is coming.”

- Spherical mirrors:
  - Object distance \( d_o \), image distance \( d_i \), focal length \( f \) and curvature radius \( R \) are positive if the corresponding object, image, focal point or center of curvature is on SLC. Otherwise, they are negative.
  - \( f = R/2 \).
  - Linear magnification: \( m \equiv h_i/h_o = -d_i/d_o \), where \( h \) is height.
  - \( d_o^{-1} + d_i^{-1} = f^{-1} \).

- Thin lenses:
  - \( f^{-1} = (n - 1)(R_1^{-1} - R_2^{-1}) \), where \( R_1 \) and \( R_2 \) are the radius of curvature of the front (SLC) surface and the back (OSLC) surface of the lens, respectively.
  - \( R > 0 \) if the center of curvature is on OSLC. It is negative otherwise.
  - Primary focal point: OSLC for converging lens, SLC for diverging lens. Secondary focal point: just the opposite.

- Thick lenses:
  - \( \frac{1}{f} = \left( \frac{n_\ell}{n_m} - 1 \right) \left[ \frac{1}{R_1} - \frac{1}{R_2} + \left( 1 - \frac{n_m}{n_\ell} \right) \frac{t}{R_1 R_2} \right] \), where \( n_\ell \) and \( n_m \) are the indices of refraction of the lens and the medium in which the lens is immersed, and \( t \) is the lens thickness.
13.1 Half-Silvered Thin Lens

[Kevin G.] Suppose you have a double convex lens made of glass \( n = 1.5 \) such that the magnitude of the radius of curvature of both sides is \( R = 42 \text{ m} \). One of the sides is painted with a layer of silver and thus acts like a spherical mirror. Assume the thin lens approximation and thus, that the lens and the mirror are essentially at the exact same position. An object lies to the left of the lens a distance \( d_o = 84 \text{ m} \) away.

(a) What is the focal point of the lens by itself? What about the mirror by itself?

(b) How many stages are there in this optical system. For each stage, state whether the light is incident from the left or from the right.

(c) For each stage, calculate the image distance and linear magnification. Make a diagram for each stage separately.

(d) Where is the final image located and what is the total linear magnification? Describe the image (i.e. is it real or virtual, upright or inverted, bigger or smaller?)

SOLUTION:

(a) Since the lens is double convex, the center of curvature of the left surface is on the right and the right surface is on the left. Thus, \( R_1 = R \) and \( R_2 = -R \), by the sign conventions, for light originating from the object on the left. From the Lensmaker equation,

\[
\frac{1}{f_l} = (n - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = (1.5 - 1)\left(\frac{1}{R} - \frac{1}{-R}\right) = \frac{1}{R} \quad \implies \quad f_l = R = 42 \text{ m}. \quad (13.1.1)
\]

For the mirror, the focal length is just half of the radius of curvature. Since this is a concave mirror, it is converging and thus has a positive focal length:

\[
f_m = R/2 = 21 \text{ m}. \quad (13.1.2)
\]

(b) There are three stages:

(i) The lens. Light goes from left to right through the lens;

(ii) The mirror. Light goes from left to right, reflects off of the mirror, and leaves going right to left;

(iii) The lens again. Light goes from right to left through the lens.

(c) Let \( d_o^{(j)} \) and \( d_i^{(j)} \) denote the object and image distances for stage \( j = 1, 2, 3 \) and so \( d_o^{(1)} \equiv d_o = 2 \text{ m} \) is the initial object distance and \( d_i^{(3)} \equiv d_i \) is the final image distance.

(i) The image distance just for the lens alone (stage 1) is

\[
\frac{1}{d_i^{(1)}} = \frac{1}{f_l} - \frac{1}{d_o^{(1)}} = \frac{1}{R} - \frac{1}{2R} = \frac{1}{2R} \quad \implies \quad d_i^{(1)} = 2R = 84 \text{ m}. \quad (13.1.3)
\]

The linear magnification of stage 1 is

\[
m_1 = -\frac{d_i^{(1)}}{d_o^{(1)}} = -\frac{84}{24} = -1. \quad (13.1.4)
\]
(ii) Since $d^{(1)}_i$ is positive, image 1 lies to the right of the lens. This is the same as object 2. Thus, object 2 lies to the right of the mirror whereas the light is actually hitting the mirror from the left! This is a virtual object with negative object distance: $d^{(2)}_o = -d^{(1)}_i = -2R = -84 \text{ m}$. So,

$$\frac{1}{d^{(2)}_i} = \frac{1}{d^{(1)}_i} - \frac{1}{d_o} = \frac{2}{R} - \frac{5}{2R} \implies d^{(2)}_i = \frac{2}{5} R = 16.8 \text{ m}. \quad (13.1.5)$$

The linear magnification of stage 2 is

$$m_2 = -\frac{d^{(2)}_i}{d^{(2)}_o} = -\frac{16.8 \text{ m}}{-84 \text{ m}} = \frac{1}{5}. \quad (13.1.6)$$

(iii) Since $d^{(2)}_i$ is positive, image 2 lies to the left of the mirror. This is the same as object 3. Thus, object 3 lies to the left of the lens whereas the light is actually entering the lens from the right! This is a virtual object with negative object distance: $d^{(3)}_o = -d^{(2)}_i = -\frac{2}{5} R = -16.8 \text{ m}$. So,

$$\frac{1}{d^{(3)}_i} = \frac{1}{d^{(2)}_i} - \frac{1}{d_o} = \frac{1}{R} - \frac{7}{2R} \implies d^{(3)}_i = \frac{2}{7} R = 12 \text{ m}. \quad (13.1.7)$$

The linear magnification of stage 3 is

$$m_3 = -\frac{d^{(3)}_i}{d^{(3)}_o} = -\frac{12 \text{ m}}{-16.8 \text{ m}} = \frac{5}{7}. \quad (13.1.8)$$

Below, virtual objects are drawn in gray. Some rules are tricky: for example, in stage 2, we can’t actually draw the light ray that starts from the object parallel to the axis and reflects off of the mirror heading towards the focal point. This is because the object is virtual and behind the mirror - there is no actual light there! In fact, the light has to go left to right. So we have to draw the light ray that looks like its headed towards the

(d) The final image is 12 m to the left of the lens. The total linear magnification is $m = m_1 m_2 m_3 = -1/7$. The image is real, inverted and smaller by a factor of 7.
Part III
Statistical Mechanics
Chapter 14

Entropy and Temperature

- Fundamental assumption of thermodynamics for Microcanonical Ensemble: a closed system is equally likely to be in any of the quantum states accessible to it.

- Zeroth law: $A \sim B$ and $B \sim C$ implies $A \sim C$ where $\sim$ means “in thermal equilibrium with”. “You must play the game.” (I don’t understand this slogan.)

- First law: Heat is a form of energy and energy is conserved. “You can’t win.” (This slogan makes more sense: you can’t get out more than you put in.)

- Second law: For a closed system: $\Delta \sigma \geq 0$, upon removal of internal constraint. “You can’t break even.”

- Third law: $\sigma \to \min$ as $\tau \to 0$. “You can’t quit the game.”

- $P(s)$ = probability for system to be in state $s$. Average: $\langle X \rangle = \sum_s X(s)P(s)$.

- Combined multiplicity of two systems: $g(s) = \sum_{s_1} g_1(s_1)g_2(s-s_1)$.

- Entropy: $\sigma(N, U) = \log g(N, U)$.

- Temperature: $1/\tau(U, N) = (\partial\sigma/\partial U)_{N, V}$.

- Stirling approximation: $N! \approx (2\pi N)^{1/2}N^Ne^{-N+\cdots}$. Most of the time, you just need to remember $\log N! \approx N \log N - N$. 

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14.1 Paramagnetism

[Kittel & Kroemer 2.2] Find the equilibrium value at temperature $\tau$ of the fractional magnetization, $M/Nm = 2\langle s \rangle /N$, of the system of $N$ spins (spin 1/2) each of magnetic moment, $m$, in a magnetic field, $B$. The spin excess is $2s$.

**SOLUTION:**

**Method 1:** Let the number of up and down spins be

$$N_\uparrow = \frac{N}{2} + s, \quad N_\downarrow = \frac{N}{2} - s.$$  \hfill (14.1.1)

Let $N = N_\uparrow + N_\downarrow$ be fixed. Then, the multiplicity function is

$$g(N, s) = \binom{N}{N_\uparrow} = \frac{N!}{N_\uparrow!N_\downarrow!} = \frac{N!}{(\frac{N}{2} + s)!(\frac{N}{2} - s)!}.$$  \hfill (14.1.2)

Using Stirling, we can write the entropy as

$$\sigma \approx N \log N - N - \left(\frac{N}{2} + s\right) \log\left(\frac{N}{2} + s\right) + \left(\frac{N}{2} + s\right) - \left(\frac{N}{2} - s\right) \log\left(\frac{N}{2} - s\right) + \left(\frac{N}{2} - s\right)$$

$$\approx -\frac{N}{2} \left\{ (1 + \frac{2s}{N}) \left[ \frac{2s}{N} - \frac{1}{2} \left(\frac{2s}{N}\right)^2 - \log 2 \right] + (1 - \frac{2s}{N}) \left[ -\frac{2s}{N} - \frac{1}{2} \left(\frac{2s}{N}\right)^2 - \log 2 \right] \right\}$$

$$= -2s^2/N + \cdots.$$  \hfill (14.1.3)

where use was made of the expansion $\log(1 + x) \approx x - \frac{1}{2} x^2$ and where $\cdots$ signify terms of higher order in $s$ or independent of $s$. This implies that the leading $s$ dependence of the multiplicity is the familiar Gaussian term: $g(N,s) \sim e^{-2s^2/N}$. I never remember Stirling’s approximation enough to really take care of the factors of $2\pi$ and such. I just work those factors out using simple logic: the sum of $g(N,s)$ over all possible $s$ must obviously give $2^N$, which is the total number of states for $N$ two-state spins. Thus, $P(s) = g(N,s)/2^N$ must be the probability of having a spin excess $2s$. Since $g(N,s)$ is Gaussian in $s$, it follows that $P(s)$ must be a normalized Gaussian. A normalized Gaussian with mean $\mu$ and standard deviation $\sigma$ is

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma^2} \exp\left[ -\frac{(x-\mu)^2}{2\sigma^2} \right].$$  \hfill (14.1.4)

It follows that the standard deviation in our case is $\sqrt{N}/2$ and so

$$\frac{g(N,s)}{2^N} = P(s) = \sqrt{\frac{2}{\pi N}} \ e^{-2s^2/N}.$$  \hfill (14.1.5)

Therefore, the entropy is

$$\sigma(N,s) = \sigma_0 - 2s^2/N,$$  \hfill (14.1.6)

where $\sigma_0 = \log g(N,0)$. In terms of energy, $U = -2smB$,

$$\sigma(N,U) = \sigma_0 - U^2/2Nm^2B^2.$$  \hfill (14.1.7)

Therefore, the temperature is

$$\frac{1}{\tau} = \left( \frac{\partial \sigma}{\partial U} \right)_N = -\frac{\langle U \rangle}{Nm^2B^2},$$  \hfill (14.1.8)

where we have assumed $N$ to be sufficiently large as to justify evaluating the RHS just at the mean value $\langle U \rangle$, since it’s highly peaked there. Thus,

$$\frac{M}{Nm} = \frac{2\langle s \rangle}{N} = -\frac{\langle U \rangle}{NmB} = \frac{mB}{\tau}.$$  \hfill (14.1.9)
Method 2: The partition function is not defined in Chapter 2, but I thought I’d show this method anyway since it’s probably how I would first approach the problem, even though the computation is a bit more difficult.

The Boltzmann factor for a state with spin excess \(s\) is \(e^{-U/\tau} = e^{2smB/\tau}\). The partition function is a sum of these for all \(2^N\) states, or, using the multiplicity function:

\[
Z = \sum_s g(N, s) e^{2smB/\tau} = \int_{-\infty}^{\infty} 2^N \sqrt{\frac{2}{\pi N}} e^{-2s^2/N} e^{2smB/\tau} ds. \tag{14.1.10}
\]

This is a simple Gaussian integral. I would first complete the square for \(s\) in the exponential, change variables to \(\xi = \sqrt{\frac{N}{2}} (s - \frac{N m B}{2 \tau})\), and use \(\int_{-\infty}^{\infty} e^{-\xi^2} d\xi = \sqrt{\pi}\) to get

\[
Z = 2^N \exp \left[ \frac{Nm^2B^2}{2\tau^2} \right]. \tag{14.1.11}
\]

A quick calculation from the definition, (14.1.10), of \(Z\), shows that

\[
\frac{1}{Z} \frac{\partial Z}{\partial \tau} = -2 \langle s \rangle m B \tau^{-2}. \tag{14.1.12}
\]

Actually calculating the LHS of (14.1.12) from (14.1.11) yields

\[
-\frac{2 \langle s \rangle m B}{\tau^2} = \frac{1}{Z} \frac{\partial Z}{\partial \tau} = -\frac{Nm^2B^2}{\tau^3}, \tag{14.1.13}
\]

from which the same result, (14.1.9), follows.

Aside: If for some reason you got stuck, use dimensional analysis and some sound physical intuition: the natural variables to work with are \(m\), \(B\) and \(\tau\). The thing we want is unitless and so it must be a function of \(\xi \equiv mB/\tau\). At low temperatures, we expect perfect alignment with the magnetic field and so the fractional magnetization should be high. At high temperatures, we expect the fractional magnetization to vanish. Thus, \(\tau\) should appear in the denominator (i.e. no inverse powers of \(\xi\) and no constant term). The simplest possible behavior is linear: \(M/Nm = \alpha m B/\tau\) for some constant \(\alpha\). You’d actually have to do the above calculations to determine that \(\alpha = 1\).
14.2 Quantum Harmonic Oscillator

[Kittel & Kroemer 2.3]

(a) Find the entropy of a set of $N$ oscillators of frequency $\omega$ as a function of the total quantum number $n$. Take $N$ to be sufficiently large as to justify $N - 1 \to N$.

(b) Let $U$ denote the total energy $n\hbar\omega$ of the oscillators. Express the entropy as $\sigma(U, N)$. Show that the total energy at temperature $\tau$ is given by the Planck distribution

$$U = \frac{N\hbar\omega}{e^{\hbar\omega/\tau} - 1}.$$  \hspace{1cm} (14.2.1)

**SOLUTION:**

(a) **Method 1:** We have to count the number of ways to distribute $n$ units of energy among $N$ oscillators. This is the familiar $n$ balls in $N$ compartments question, which we can state in terms of balls and dividers (à la Schroeder). We need $N - 1$ to dividers to divide out $N$ partitions on a straight line and $n$ balls to place in them. We line them up starting with the balls in the first compartment, followed by a vertical divider and so on. We have $n + N - 1$ total symbols (balls and dividers). The number of possible arrangements is the number of ways we can choose $n$ of these symbols to be balls: $\binom{n + N - 1}{n}$. We are told to say $N - 1 \approx N$.

So, using Stirling,

$$\sigma \approx N \left[ \left(1 + \frac{n}{N}\right) \log \left(1 + \frac{n}{N}\right) - \frac{n}{N} \log \frac{n}{N} \right].$$  \hspace{1cm} (14.2.2)

**Method 2:** We can also view this problem as counting the number of ways of writing some non-negative integer, $n$, as the sum of $N$ ordered non-negative integers. Consider the infinite polynomial

$$(1 - x)^{-N} = (1 + x + x^2 + \cdots)^N = \sum_n g(N, n)x^n.$$  \hspace{1cm} (14.2.3)

Pick a particular $n$ and examine the ways in which $x^n$ arises from the LHS. The first set of possibilities is when all of $n$ factors of $x$ come from just one of the $N$ factors of $1/(1 - x)$. The next is when $n - 1$ come from one and one from another, and so on. Thus, the coefficient of $x^n$, which we have called $g(N, n)$ is precisely the number of ways of writing $n$ as the sum of $N$ ordered non-negative integers. That is, it is the multiplicity function we want.

To pick out $g(N, n)$ from $(1 - x)^{-N}$, differentiate the latter $n$ times, divide by $n!$ and then set $x = 0$ (to get rid of all higher powers of $x$ still remaining. Doing so exactly yields the result of method 1.

(b) Just write $n = U/\hbar\omega$:

$$\sigma(U, N) = N \left[ \left(1 + \frac{U}{N\hbar\omega}\right) \log \left(1 + \frac{U}{N\hbar\omega}\right) - \frac{U}{N\hbar\omega} \log \frac{U}{N\hbar\omega} \right].$$  \hspace{1cm} (14.2.4)

The temperature is

$$\frac{1}{\tau} = \left( \frac{\partial \sigma}{\partial U} \right)_N = \frac{1}{\hbar\omega} \log \left(1 + \frac{N\hbar\omega}{U}\right).$$  \hspace{1cm} (14.2.5)

Solving for $U$ yields the Planck distribution (14.2.1).
Boltzmann Distribution and Helmholtz Free Energy

- Canonical ensemble: $N$ constant, system can exchange energy with reservoir. $P(\epsilon_s) = e^{-\epsilon_s/\tau} / Z$ where $Z = \sum_s e^{-\epsilon_s/\tau}$.

- Pressure: $p = - (\partial U / \partial V)_\sigma = \tau (\partial \sigma / \partial V)_\sigma$.

- Helmholtz free energy: $F = U - \tau \sigma$ is a minimum in equilibrium for a system held at constant $\tau$ and $V$.

- $U$ good for constant $\sigma$, whereas $F$ good for constant $\tau$.

- $F = - \tau \log Z$ (both sides satisfy $- \tau^2 \partial_\tau (F / \tau) = U$.)

- Ideal monoatomic gas (no spin; identical particles): $Z_N = (n_Q V)^N / N!$ where $n_Q = (M\tau / 2\pi \hbar^2)^{3/2}$. Classical condition: $n/n_Q << 1$.

- Ideal gas law: $pV = N\tau$.

- Ideal gas entropy (Sackur-Tetrode): $\sigma = N [\log (n_Q/n) + \frac{5}{2}]$.

- Monoatomic ideal gas heat capacity at constant volume: $C_V = 3N/2$.

- Reversible process: always infinitesimally close to equilibrium state.

- Remember: However Very Good Students Fail Pretty Easy Tests.
15.1 Free Energy of a Harmonic Oscillator

[Kittel & Kroemer 3.3] In Problem 14.2, we worked out the entropy, $\sigma(U, N)$, and internal energy, $U(N, \tau)$, of a set of $N$ oscillators of frequency $\omega$ and total quantum number $n$.

(a) Write $\sigma(N, \tau)$ by substituting in $U(N, \tau)$ into $\sigma(U, N)$.

(b) Do this problem a different way: calculate the free energy, $F(N, \tau)$ and use this to calculate the entropy.

SOLUTION:

(a) First, let us write (14.2.5) as

$$\frac{1}{\tau} = \frac{1}{\hbar \omega} \left[ \log(1 + \frac{U}{N\hbar \omega}) - \log \frac{U}{N\hbar \omega} \right].$$

(15.1.1)

Then, write the entropy, (14.2.4) as

$$\sigma(U, N) = N \log(1 + \frac{U}{N\hbar \omega}) + \frac{U}{\hbar \omega} \left[ \log(1 + \frac{U}{N\hbar \omega}) - \log \frac{U}{N\hbar \omega} \right].$$

(15.1.2)

Combining these two, along with $\frac{N\hbar \omega}{U} = e^{\hbar \omega/\tau} - 1$, yields

$$\sigma(N, \tau) = N \left[ \frac{\hbar \omega/\tau}{e^{\hbar \omega/\tau} - 1} - \log(1 - e^{-\hbar \omega/\tau}) \right].$$

(15.1.3)

(b) The partition function for a single harmonic oscillator is

$$Z_1 = \sum_{s} e^{-s\hbar \omega/\tau} = \frac{1}{1 - e^{-\hbar \omega/\tau}}.$$

(15.1.4)

For $N$ oscillators, usually, the oscillators are in a crystal where their position in the crystal distinguishes them. Thus, we will consider the oscillators distinguishable. In any case, this makes no difference for the entropy. Then, the total partition function is $Z = Z_1^N$ and the free energy is

$$F = -\tau \log Z = N \tau \log \left(1 - e^{-\hbar \omega/\tau} \right).$$

(15.1.5)

The entropy is in agreement with part (a):

$$\sigma = - \left( \frac{\partial F}{\partial \tau} \right)_{V, N} = N \left[ \frac{\hbar \omega/\tau}{e^{\hbar \omega/\tau} - 1} - \log(1 - e^{-\hbar \omega/\tau}) \right].$$

(15.1.6)
15.2 Zipper Problem

[Kittel & Kroemer 3.7] A zipper has \( N \) links; each link has a state in which it is closed with energy 0 and a state in which it is open with energy \( \epsilon \). We require, however, that the zipper can only unzip from the left end, and that the link number \( s \) can only open if all links to the left \((1, 2, \ldots, s - 1)\) are already open.

(a) Calculate the partition function.

(b) In the limit \( \epsilon >> \tau \), find the average number of open links. The model is a simplified model of the unwinding of two-stranded DNA molecules.

SOLUTION:

(a) There is only one state for any given number, \( n \), of open links. Thus, the partition function is

\[
Z = \sum_{n=0}^{N} e^{-\beta n\epsilon} = \sum_{n=0}^{\infty} e^{-\beta n\epsilon} - \sum_{n=N+1}^{\infty} e^{-\beta n\epsilon} = \frac{1 - e^{-\beta (N+1)\epsilon}}{1 - e^{-\beta \epsilon}}.
\]

(b) In the limit \( \beta \epsilon >> 1 \), we can neglect the \( e^{-\beta (N+1)\epsilon} \) term in \( Z \) and then the average number of open links is

\[
\langle n \rangle = -(1/\epsilon) \partial_{\beta} \log Z \approx \frac{1}{e^{\beta \epsilon} - 1},
\]

which is just the Planck distribution!
15.3 Magnetic Cooling

[Fall 2007 Modern (Morning), Problem 6] Consider a system of $N_s$ distinguishable spin-$\frac{1}{2}$ particles in a magnetic field $B$ at temperature $\tau = kT$. Each spin has a magnetic moment $\mu$ and its energy in the magnetic field is $\epsilon_{\pm} = \mp \mu B$, depending on whether it points along or opposite the magnetic field.

(a) Write down the partition function of each spin and probabilities of it pointing along or opposite to the magnetic field.

(b) What is the total moment $M$ of the system? What constitutes “high temperature”? In this regime, show that the magnetic susceptibility $\chi = M/B$ behaves as $\chi = N_s \mu^2/\tau$ (Curie’s law).

(c) At high temperature, show that the entropy behaves as

$$\sigma_s = \frac{S_s}{k} = N_s \left( \log 2 - \frac{\mu^2 B^2}{2\tau^2} \right). \tag{15.3.1}$$

(d) Couple the spins to a phonon gas in an insulating crystal at the same temperature. In the Debye model for lattice vibrations, the entropy of the phonon system is

$$\sigma_p = \frac{S_p}{k} = \frac{4}{3} \pi^4 N_p \left( \frac{\tau}{\tau_D} \right)^3, \tag{15.3.2}$$

where $N_p$ is the number of atoms in the lattice and $\tau_D$ is the Debye temperature. Isolate the system and reduce the magnetic field to zero. What happens? Compute the final temperature $\tau_f$.

**SOLUTION:**

(a) $Z_1 = e^{\mu B/\tau} + e^{-\mu B/\tau} = 2 \cosh(\mu B/\tau)$. $P(\epsilon_+) = e^{\mu B/\tau}/Z$ and $P(\epsilon_-) = e^{-\mu B/\tau}/Z$.

(b) The numbers of spin up and down are $N_\uparrow = P(\epsilon_+) N_s$ and $N_\downarrow = P(\epsilon_-) N_s$. Thus, the total magnetic moment is

$$M = [P(\epsilon_+) - P(\epsilon_-)] N_s \mu = N_s \mu \tanh(\mu B/\tau). \tag{15.3.3}$$

At small $\mu B/\tau$, $\tanh(\mu B/\tau) \approx \mu B/\tau$ and thus,

$$\chi = M/B = N_s \mu (\mu/\tau) = N_s \mu^2/\tau. \tag{15.3.4}$$

(c) Define $x \equiv \mu B/\tau$. Since the particles are distinguishable, $Z = Z_1^{N_s}$ (no $1/N_s!$). Thus, the free energy is

$$F = -\tau \log Z = -N_s \tau [\log 2 + \log \cosh x]. \tag{15.3.5}$$

The entropy is given by

$$\sigma_s = -\left( \frac{\partial F}{\partial \tau} \right)_V = N_s \log 2 + N_s \log \cosh x + N_s \tau \frac{1}{\cosh x} \sinh x \frac{dx}{\tau}$$

$$= N_s \log 2 + N_s \log \cosh x - N_s x \tanh x, \tag{15.3.6}$$

where use was made of $dx/d\tau = -x/\tau$. 

Now, we expand in small $x$: $\cosh x \approx 1 + \frac{1}{2}x^2$ and $\log(1 + \epsilon) \approx \epsilon$ implies $\log \cosh x \approx \frac{1}{2}x^2$. Of course, $\tanh x \approx x$. Thus, we get

$$\sigma_s \approx N_s \log 2 + \frac{1}{2}N_s x^2 - N_s x^2 = N_s \left[ \log 2 - \frac{1}{2}x^2 \right]. \tag{15.3.7}$$

which is precisely the desired result.

(d) Since the process is reversible, the sum $\sigma_s + \sigma$ does not change, but, since $\sigma_s$ increases as $B \to 0$, then $\sigma$, and thus $T$, decreases. The result is

$$\tau_f = \tau_i \left[ 1 - \frac{5}{8\pi^2} \frac{N_s}{N} \left( \frac{\mu B}{\tau_i} \right)^2 \left( \frac{\tau_D}{\tau_i} \right)^3 \right]^{1/3}. \tag{15.3.8}$$
15.4 Free Relativistic Gas

[Fall 2007 Modern (Afternoon), Problem 5] Consider a gas contained in volume $V$ at temperature $T$. The gas is composed of $N$ distinguishable non-interacting particles of zero mass, so that energy $E$ and momentum $p$ of the particles are related by $E = pc$. The number of single-particle energy states in the range $p$ to $p + dp$ is $4\pi V p^2 dp/h^3$.

(a) Use the partition function to find the equation of state and the internal energy of the gas.

(b) Compare with an ordinary gas.

SOLUTION:

(a) The single particle partition function is

$$Z_1 = \int_0^\infty e^{-\beta pc} \frac{4\pi V}{h^3} p^2 dp = \frac{8\pi V}{(\beta hc)^3}.$$  \hspace{1cm} (15.4.1)

Since the particles are distinguishable, the total partition function is $Z = Z_1^N$. We can now compute the internal energy:

$$U = \langle E \rangle = N \langle p \rangle_1 c = -N \partial_\beta \log Z_1 = 3NkT.$$  \hspace{1cm} (15.4.2)

You might think of finding the pressure via $P = -(\partial U/\partial V)_{S,N}$, but the problem is that we do not have $U$ explicitly in terms of $S$ and so it will be hard to know the effect of keeping $S$ constant. But, we can find the free energy:

$$F = -kT \log Z = -NkT \log[8\pi V(kT/hc)^3],$$  \hspace{1cm} (15.4.3)

which is explicitly in terms of $N$ and $T$, and so we can compute the pressure:

$$P = -(\partial F/\partial V)_{T,N} = NkT/V,$$  \hspace{1cm} (15.4.4)

which is just the same old ideal gas law.

(b) The equation of state is exactly the same. However, the internal energy is twice that of an ordinary monoatomic ideal gas, which would have $U = 3NkT/2$. 

15.5 Energy Deviation from the Mean

[Fall 2007 Modern (Afternoon), Problem 6; Kittel & Kroemer 3.4] The average energy of a system in thermal equilibrium is $\langle E \rangle$.

(a) Use the partition function to prove that the mean square deviation of energy from the mean, $\langle (E - \langle E \rangle)^2 \rangle$ is given by $kT^2C_V$, where $C_V$ is the heat capacity of the entire system at constant volume.

(b) Use this result to show that the energy of a macroscopic system may ordinarily be considered constant when the system is in thermal equilibrium.

**SOLUTION:**

(a) The internal energy is $U = \langle E \rangle = -\frac{1}{Z} \frac{\partial Z}{\partial \beta}$. The average of the square of the energy is $\langle E^2 \rangle = \frac{1}{Z} \frac{\partial^2 Z}{\partial \beta^2}$. Thus, variance is

$$\sigma^2 \equiv \langle (E - \langle E \rangle)^2 \rangle = \langle E^2 \rangle - \langle E \rangle^2 = \frac{1}{Z} \frac{\partial^2 Z}{\partial \beta^2} - \left( \frac{1}{Z} \frac{\partial Z}{\partial \beta} \right)^2. \quad (15.5.1)$$

We recognize that we can write this as

$$\sigma^2 = \frac{1}{Z} \frac{\partial}{\partial \beta} \left( \frac{1}{Z} \frac{\partial Z}{\partial \beta} \right) = -\frac{\partial U}{\partial \beta} = kT^2 \frac{\partial U}{\partial T} = kT^2C_V. \quad (15.5.2)$$

(b) What matters is the fractional variance (or standard deviation). For $U = \langle E \rangle \sim NkT$ (up to order 1 numerical prefactors), $C_V \sim Nk$. Thus,

$$\frac{\langle (E^2) - \langle E \rangle^2 \rangle}{\langle E^2 \rangle} \sim \frac{N(kT)^2}{(NkT)^2} = \frac{1}{N} \xrightarrow{N \to \infty} 0. \quad (15.5.3)$$
Chapter 16

Thermal Radiation and Planck Distribution

- Planck distribution: thermal average of photons in a cavity mode of frequency $\omega$ is
  \[ \langle s \rangle = \frac{1}{(e^{h\omega/\tau} - 1)^{-1}}. \]

- Stefan-Boltzmann law: $U/V = (\pi^2/15h^3c^3)\tau^4$.

- Photon entropy: $\sigma = (4\pi^2V/45)(\tau/\hbar c)^3$.

- Planck radiation law: $u_\omega = \frac{h}{2\pi^2}\omega^3(e^{h\omega/\tau} - 1)^{-1}$ (integrate over $\omega$ to get $U/V$).

- Radiant energy flux density: $J_U = [eU/V] \times$ geometric factor. For a circular hole, the geometric factor is $1/4$ and $J_U = \sigma_B T^4$ where the Stefan-Boltzmann constant is $\pi^2k^4/60\hbar^3c^2$.

- Debye: $U = (3\pi^4N/5\vartheta^3)\tau^4$ where $\vartheta = k\theta = \hbar v(6\pi^2N/V)^{1/3}$ and $v$ is the propagation speed of phonons.

- Debye heat capacity: $C_V = (12\pi^4N/5)(\tau/\vartheta)^3$. 

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16.1 Pressure of Thermal Radiation

[Kittel & Kroemer 4.6] Show for a photon gas that:

(a) \( p = -\left(\frac{\partial U}{\partial V}\right)_s = -\sum_j s_j h \omega_j / dV \), where \( s_j \) is the number of photons in the mode \( j \).

(b) \( d\omega_j / dV = -\omega_j / 3V \) for isotropic volume changes.

(c) \( p = U / 3V \). Thus, the radiation pressure is a third of the energy density.

(d) Compare the pressure of thermal radiation with the kinetic pressure of a gas of H atoms at a concentration of 1 mole/cm\(^3\) characteristic of the sun. At what temperature (roughly) are the two pressures equal?

**SOLUTION:**

(a) Assuming adiabatic volume change, \( (\partial s_j / \partial V)_s = 0 \) (i.e. the occupation numbers will not change). Then, noting that \( \omega_j \) depends only on the dimensions of the cavity and thus its partial derivative can be written as a regular derivative, we have

\[
p = -\left(\frac{\partial}{\partial V}\right)_s \sum_j s_j h \omega_j = -\sum_j s_j h (d\omega_j / dV) \tag{16.1.1}
\]

(b) The frequency is \( \omega_j = j\pi c / L = j\pi c V^{-1/3} \) and so

\[
d\omega_j / dV = -\frac{1}{3} j\pi c V^{-4/3} = -\omega_j / 3V \tag{16.1.2}
\]

(c) Combining the previous two equations yields

\[
p = -\sum_j s_j h (d\omega_j / dV) = \sum_j s_j h \omega_j / 3V = U / 3V \tag{16.1.3}
\]

(d) Using Stefan-Boltzmann, \( p = -\frac{4\sigma_B}{3c} T^4 \). The H gas pressure is \( p = NkT \). Setting these equal yields

\[
T = \left(\frac{3Nk c}{4\sigma_B}\right)^{1/3} \approx 3.2 \times 10^7 \text{ K} \tag{16.1.4}
\]
16.2 Isentropic Expansion of Photon Gas

[Kittel & Kroemer 4.18] Consider the gas of photons of the thermal equilibrium radiation in a cube of volume $V$ at temperature $\tau$. Let the cavity volume increase; the radiation pressure performs work during the expansion, and the temperature of the radiation will drop. From the result for the entropy, we know that $\tau V^{1/3}$ is constant in such an expansion.

(a) Assume that the temperature of the cosmic black-body radiation was decoupled from the temperature of the matter when both were at 3000 K. What was the radius of the universe at that time, compared to now, when the CMB temperature is $\sim 3$ K? If the radius has increased linearly with time, at what fraction of the present age of the universe did the decoupling take place?

(b) Show that the work done by the photons during the expansion is given by the expression $W = (\pi^2/15\hbar^3c^3)V_i\tau_i^3(\tau_i - \tau_f)$. The subscripts $i$ and $f$ refer to the initial and final states.

SOLUTION: [Thanks to Victoria]

(a) You ought to derive the fact that $\tau V^{1/3}$ is constant for isentropic expansion. Doing Problem 4.7 (p.112) may help. Since $V \propto R^3$, we know that $\tau R$ is constant. Hence,

$$\frac{R_i}{R_f} = \frac{\tau_f}{\tau_i} \approx \frac{3 \text{ K}}{3000 \text{ K}} = 10^{-3} \tag{16.2.1}$$

The decoupling took place at the same fraction, $10^{-3}$, of the present age of the universe, if $R$ grew linearly with time.

(b) Since the process is isentropic, the work is just the difference in internal energies (i.e. there is no heat term). From Stefan-Boltzmann, $U = (\pi^2/15\hbar^3c^3)V\tau^4$. Thus,

$$W = U_i - U_f = \frac{\pi^2}{15\hbar^3c^3}[(V_i\tau_i^3)\tau_i - (V_f\tau_f^3)\tau_f] = \frac{\pi^2}{15\hbar^3c^3}V_i\tau_i^3(\tau_i - \tau_f) \tag{16.2.2}$$

where we used the fact that $V\tau^3$ is constant.
16.3 Radiation, the Sun and the Earth

[Fall 2008 Modern (Morning), Problem 2]

(a) Consider the thermal radiation in a cavity of volume $V$ at temperature $T$. Show that the energy density of the photons in the cavity is $U/V = AT^4$, where $A$ is a constant that depends only on fundamental physical constants. Show that the flux of radiant energy from a small hole in the wall of the cavity (named black body radiation) is given by $J_U = BT^4$, with $B$ being another constant. [Note: You don’t have to work out the constants $A$ and $B$.]

(b) Let us assume that the sun and the earth are perfect black-body radiators and that there is no source of energy other than the sun. Show that at steady-state, the temperature on the earth’s surface is linearly proportional to the sun’s surface temperature. Find the expression for the proportionality constant.

(c) To reduce global warming, it has been suggested recently that one can place a giant solar screen (at the so-called Lagrange point) between the sun and the earth to reduce the solar flux arriving on earth. Suppose that one is able to construct such a screen to reduce the solar flux to earth by 5%. By how much (in K) will the average temperature of the earth’s surface be reduced? Assume the value for the earth’s surface temperature without the screen to be 280 K.

**SOLUTION:**

(a) The average energy of the photons in the (presumed cubic) cavity is

$$U = 2 \sum_n \frac{h \omega_n}{e^{h \omega_n / kT} - 1},$$

(16.3.1)

where $\omega_n = n \pi c / L$ and $n = \sqrt{n_x^2 + n_y^2 + n_z^2}$. Since we are not asked to calculate the constant, the factor of 2 is immaterial, but it’s there since each photon has two independent polarizations.

First, we change the sum to $\int dn_x \, dn_y \, dn_z$. The integers $n_i$ are non-negative. But, we can extend the integral over all space and just divide by 8 (to restrict to the positive octant). Hence,

$$U = \frac{2}{8} \int_0^\infty 4\pi n^2 \frac{\pi hc/L}{e^{\pi hc/L kT} - 1} \, dn.$$  

(16.3.2)

Changing variables to $x = \pi hc/L \tau$ gives

$$\frac{U}{V} = \left( \frac{\pi}{k} \right)^3 \int_0^\infty \frac{x^3 \, dx}{e^x - 1} \, T^4$$

(16.3.3)

The flux density is proportional to $U/V$ by geometric factors (and a speed of light). Thus, if $U/V \sim T^4$, then $J_U \sim T^4$ as well.

(b) The power radiating from the sun is $P_s = 4\pi R_s^2 BT_s^4$, where $R_s$ is the sun’s radius. The intensity at earth’s orbit is $I_e = P_s/(4\pi R_{es}^2) = (R_s/R_{es})^2 BT_s^4$, where $R_{es}$ is the earth-sun distance. The power intercepted by the earth is $P_e = I_e \pi R_e^2 = \pi (R_e R_s/R_{es})^2 BT_s^4$. Note: $\pi R_e^2$ is the area of the circular cross-section of the earth as viewed from the sun. At equilibrium, all of this absorbed power is reradiated, and so $\pi (R_e R_s/R_{es})^2 BT_s^4 = 4\pi R_e^2 BT_e^4$ and so

$$T_e = T_s \sqrt{R_s/2R_{es}}.$$  

(16.3.4)
(c) The new $P_e$ would be $P'_e = 0.95 P_e$, or the new temperature, $T'_e = 0.95^{1/4} T_e = 276.4$ K. Thus, the average temperature is decreased by $3.6$ K.
16.4 Heat Shields

[Kittel & Kroemer 4.8 & 4.19] A black (nonreflective) plane at temperature $T_u$ is parallel to a black plane at temperature $T_l$. The net energy flux density in vacuum between the two planes is $J_U = \sigma_B(T_u^4 - T_l^4)$, where $\sigma_B$ is the Stefan-Boltzmann constant. A third black plane is inserted between the other two and is allowed to come to a steady state temperature, $T_m$. Find $T_m$ in terms of $T_u$ and $T_l$, and show that the net energy flux density is cut in half because of the presence of this plane. This is the principle of the heat shield and is widely used to reduced radiant heat transfer.

Show that if the middle plane has reflectivity, $r$, then the previous result for the flux density (when $r = 0$) is simply multiplied by $1 - r$.

SOLUTION:

Below, we draw the sources of flux density in each region:

When the middle plane is in equilibrium, the flux in some direction, say to the left, on either side of the middle plane will be the equal. The leftward flux on the left of the middle plane is $J_m - J_l$, whereas the leftward flux to the right of the middle plane is $J_u - J_m$. Setting these equal yields $J_m = (J_u + J_l)/2$. Therefore,

$$\sigma_B T_m^4 = \frac{1}{2} \sigma_B (T_u^4 + T_l^4) \quad \Rightarrow \quad T_m = \left[ \frac{1}{2} (T_u^4 + T_l^4) \right]^{1/4}. \quad (16.4.1)$$

The flux incident on the plane at $T_l$ is

$$J_U = J_m - J_l = \frac{1}{2} (J_u + J_l) - J_l = \frac{1}{2} (J_u - J_l) = \frac{1}{2} \sigma_B (T_u^4 - T_l^4), \quad (16.4.2)$$

which is, indeed, half of its value without the heat shield.

If the reflectivity is $r$, call the corresponding fluxes due to the middle plane the same symbol, but with primes. Then on the right side, in addition to $J_m'$, we have $rJ_u$, and on the left side, in addition to $J_m'$, we have $rJ_l$. Setting the leftward flux on both sides equal to each other and solving for $J_m'$ yields $J_m' = (1 - r)\frac{1}{2} (J_u + J_l) = (1 - r)J_m$, where $J_m$ is the corresponding value for $r = 0$. Hence, the flux incident on the lower temperature plane is

$$J_U' = J_m' + rJ_l - J_l = (1 - r) (J_m - J_l) = (1 - r)J_U. \quad (16.4.3)$$
16.5 Cylindrical Heat Shields

Kevin G. A long narrow cylindrical chamber has radius 10 cm and length 2 m and is in thermal equilibrium with its liquid nitrogen contents at 67 K. The chamber is encased in a larger coaxial cylinder of radius 20 cm and length 2 m and the space between the two cylinders is evacuated. The outer casing is always in thermal equilibrium with the environment at room temperature, \( \sim 300 \) K. At standard pressure, nitrogen boils at \( \sim 77 \) K.

(a) How long will it take for the nitrogen to heat up to its boiling temperature? [Liquid nitrogen density is \( \sim 0.807 \) g/cm\(^3\) and the specific heat is \( \sim 2.042 \) J/gK.]

(b) 99 perfectly emissive concentric cylindrical heat shields are placed at equal radial intervals between the inner and outer casing. Repeat part (a) in this case.

**SOLUTION:**

(a) The mass of the nitrogen is \( m = \rho \pi r^2 \ell = 5.0705 \times 10^4 \) g, where \( r = 10 \) cm is the radius of the inner chamber and \( \ell = 200 \) cm its length. Therefore, the heat capacity is \( C = mc = 1.0354 \times 10^5 \) J/K. The radiant energy flux from the outer casing at the inner casing is \( J_0(r) = \frac{2\pi r \sigma_B T_0^4}{\pi r} = \frac{r_0 \sigma_B T_0^4}{r} \), where \( r_0 = 20 \) cm is the outer radius and \( T_0 = 300 \) K is the outer temperature. Hence, the total flux inward at the inner casing is \( J = \sigma_B (\frac{r_0}{r} T_0^4 - T^4) \).

The incoming power is \( \dot{Q} = 2\pi r \ell J \).

Assume that all of the incoming energy is used to heat the nitrogen; that is, a negligible amount is used to heat up the inner casing itself. Then,

\[
\frac{dT}{dt} = \frac{1}{C} \frac{dQ}{dt} = \frac{2\pi r \sigma_B}{C} \left( \frac{r_0}{r} T_0^4 - T^4 \right) = -\alpha T^4 + \beta T_0^4, \tag{16.5.1}
\]

where \( \alpha = 6.882 \times 10^{-9} \) 1/K\(^3\)s and \( \beta = 1.3764 \times 10^{-8} \) 1/K\(^3\)s.

The homogeneous equation may be written \( \frac{dT}{dt} T^{-3} = \frac{2\pi r \sigma_B}{C} \frac{d}{dt} \frac{1}{T^3} = 3\alpha \), whose general solution is \( T = (3\alpha t + A)^{-1/3} \), where \( A \) is a constant. A specific solution to the inhomogeneous equation is \( T = (r_0/r)^{1/4} T_0 \). Let \( T_i = 67 \) K be the initial temperature of the inner chamber. The solution is

\[
T(t) = -\left\{ [(r_0/r)^{1/4} T_0 - T_i]^{-3} - 3\alpha t \right\}^{-1/3} + (r_0/r)^{1/4} T_0. \tag{16.5.2}
\]

Set this equal to \( T_f = 77 \) K and solve for \( t \) to get

\[
t = \frac{1}{3\alpha} \left\{ [(r_0/r)^{1/4} T_0 - T_f]^{-3} - [(r_0/r)^{1/4} T_0 - T_i]^{-3} \right\} \approx 0.22 \text{ s} \tag{16.5.3}
\]

(b) Let \( r_0 \) be the outer radius, \( r = r_{100} \) the inner radius and \( r_n \) in between. The power between \( r_n \) and \( r_{n+1} \) is \( Q_{in}^{(n)} = 2\pi r \nu \ell J_n - 2\pi r_{n+1} \ell J_{n+1} \). Assuming that the process is always close to equilibrium, \( Q_{in}^{(n)} = Q_{in} \) are all the same. Add up all these equations to get \( 2\pi \ell (r_0 J_0 - r_{100} J_{100}) = 100 \dot{Q}_m \). Thus, heat is transferred 100 times slower: \( t \approx 22 \text{ s} \).
Chemical Potential and Gibbs Distribution

- Chemical potential: \( \mu (\tau, V, N) \equiv (\partial F/\partial N)_{\tau,V} \) or \( \mu (\sigma, V, N) \equiv (\partial U/\partial N)_{\sigma,V} = -\tau (\partial \sigma/\partial N)_{U,V} \), where \( \tau(\sigma, V, N) \equiv (\partial U/\partial \sigma)_{V,N} \).

- Diffusive equilibrium means \( \mu_1 = \mu_2 \).

- Particle flow: from high to low chemical potential.

- Free ideal gas: \( \mu = \tau \log(n/n_Q) \) where \( n = N/V \) and \( n_Q = (M\tau/2\pi\hbar^2)^{3/2} \).

- If external potentials exist (e.g. electrons in a circuit connected to a battery, gas particles in the presence of gravity): \( \mu = \mu_{\text{int}} + \mu_{\text{ext}} \).

- Thermodynamic identity: \( dU = \tau d\sigma - p dV + \mu dN \).

- Gibbs factor: \( e^{(\mu N - \epsilon_s)/\tau} \).

- Grand-canonical partition function: \( Z = \sum_{\sigma,N} e^{(\mu N - \epsilon_s)/\tau} \).
17.1 Ascent of Sap in Trees

[Kittel & Kroemer 5.12] Find the maximum height to which water may rise in a tree under the assumption that the roots stand in a pool of water and the uppermost leaves are in air containing water vapor at a relative humidity \( r = 0.9 \). The temperature is 25°C. If the relative humidity is \( r \), the actual concentration of water vapor in the air at the uppermost leaves is \( r n_0 \), where \( n_0 \) is the concentration in the saturated air that stands immediately above the pool of water. [Note: treat the water vapor as an ideal gas.]

**SOLUTION:**

The internal chemical potential of an ideal gas is \( \mu_{\text{int}} = \tau \log(n/n_Q) \), where \( n \) is the volume density of water vapor molecules and \( n_Q = (M\tau/2\pi \hbar^2)^{3/2} \) is the quantum density (in 3D). At a relative humidity, \( r \), the concentration is \( n = r n_0 \). Let \( r \) vary as some function of the height \( h \) above the ground, \( r(h) \), with \( r(0) = 1 \) and \( r \equiv r(h) \). Then, the internal chemical potential of the water vapor as a function of height above the ground is

\[
\mu_{\text{int}}(h) = \tau \log \left[ r(h) n_0 / n_Q \right].
\] (17.1.1)

This is the chemical potential driving the evaporation of water off of the leaves at height \( h \). There is also a chemical potential (external) that drives the raising of water up the height \( h \), and that is simply gravitational:

\[
\mu_{\text{ext}}(h) = mgh.
\] (17.1.2)

The total chemical potential is the sum of these two and at diffusive equilibrium, \( \Delta \mu_{\text{tot}} = \Delta \mu_{\text{int}} + \Delta \mu_{\text{ext}} = 0 \). Hence, defining \( \Delta \mu_{\text{ext}} = \mu_{\text{ext}}(h) - \mu_{\text{ext}}(0) \) and similarly for \( \mu_{\text{int}} \), we find

\[
mgh = \Delta \mu_{\text{ext}} = -\Delta \mu_{\text{int}} = -\tau \left[ \log(r n_0/n_Q) - \log(n_0/n_Q) \right] = -\tau \log r.
\] (17.1.3)

Solving for \( h \) and plugging in numbers yields

\[
h = \frac{\tau}{mg} \log(1/r) \approx 1.5 \text{ km},
\] (17.1.4)

where we have used \( m = 18u \) for water.
17.2 Multiple Binding of Oxygen

[Kittel & Kroemer 5.14] A hemoglobin molecule can bind four O$_2$ molecules. Assume that $\epsilon$ is the energy of each bound O$_2$, relative to O$_2$ at rest at infinite distance. Let $\lambda = e^{\mu/\tau}$ denote the absolute activity of the free O$_2$ (in solution).

(a) What is the probability that one and only one O$_2$ is adsorbed on a hemoglobin molecule? Sketch the result qualitatively as a function of $x = \lambda e^{-\epsilon/\tau}$.

(b) What is the probability that four and only four O$_2$ are adsorbed? Sketch this result as well.

**SOLUTION:**

The Gibbs partition function for one site is $Z_1 = 1 + \lambda e^{-\epsilon/\tau}$. For four independent sites, which are necessarily distinguishable, $Z_4 = Z_1^4 = 1 + 4\lambda e^{-\epsilon/\tau} + \cdots + \lambda^4 e^{-4\epsilon/\tau}$.

(a) $P_1 = \frac{1}{Z_4} 4\lambda e^{-\epsilon/\tau} = 4x/(1 + x)^4$, where $x = \lambda e^{-\epsilon/\tau}$.

(b) $P_4 = \frac{1}{Z_4} \lambda^4 e^{-4\epsilon/\tau} = [x/(1 + x)]^4$.

![Figure 17.1: Probabilities for 1 or 4 adsorbed O$_2$ in hemoglobin.](image)

**Aside:** A question came up as to why the first term in the grand canonical partition function is 1 and not $\lambda = e^{\mu/\tau}$, where the latter is $e^{-(\epsilon-\mu)/\tau}$ after plugging in $\epsilon = 0$ for zero adsorption. But, remember that the Gibbs factor is $e^{-(\epsilon-\mu N)/\tau}$ and for zero adsorption, $N = 0$, so we get 1 for the zero adsorption term.
17.3 Hydrogen Storage

[D.H. Lee Sp ’09 MT 1] Consider the hydrogen storage problem wherein hydrogen gas (assumed ideal) is in an encasing with a piston serving as a pressure reservoir (i.e. maintaining constant pressure) and enclosed by a heat bath (i.e. maintaining constant temperature). The bottom of the box is lined with a substrate onto which hydrogen may be adsorbed with each adsorption site able to bind one molecule in one of two states. The first state has binding energy \( -\epsilon_0 \) and the second state has binding energy \( -\epsilon_0/2 \). Derive the coverage (fraction of adsorbed sites) as a function of temperature and pressure.

**SOLUTION:**

The Grand Canonical partition function for a single adsorption site is

\[
Z_1 = 1 + \lambda e^{\epsilon_0/\tau} + \lambda e^{\epsilon_0/2\tau},
\]

(17.3.1)

where \( \lambda = e^{\mu/\tau} \) is the absolute activity.

The coverage is

\[
f = \frac{\lambda e^{\epsilon_0/\tau} + \lambda e^{\epsilon_0/2\tau}}{1 + \lambda e^{\epsilon_0/\tau} + \lambda e^{\epsilon_0/2\tau}} = \frac{1 + e^{-\epsilon_0/2\tau}}{\lambda^{-1} e^{-\epsilon_0/\tau} + 1 + e^{-\epsilon_0/2\tau}}.
\]

(17.3.2)

The chemical potential of a free ideal gas is

\[
\mu = \tau \log(n/n_Q),
\]

where \( n_Q = (M\tau/2\pi\hbar^2)^{3/2} \) and \( n = N/V = p/\tau \). Thus, \( \lambda = e^{\mu/\tau} = p/\tau n_Q \):

\[
f = \frac{1 + e^{-\epsilon_0/2\tau}}{(n_Q \tau/p)e^{-\epsilon_0/\tau} + 1 + e^{-\epsilon_0/2\tau}} = \frac{p(1 + e^{-\epsilon_0/2\tau})}{p_0 + p(1 + e^{-\epsilon_0/2\tau})},
\]

(17.3.3)

where \( p_0 = n_Q \tau e^{-\epsilon_0/\tau} \).

Compare this to Eqn. (71) K & K p.143, where there is only one adsorption state with energy \( \epsilon \). We see that there is just a correction term \( p \to p(1 + e^{-\epsilon_0/2\tau}) \).
17.4 Multiple Adsorption Sites and Energies

[Kevin G.] A box contains a mixture of ideal gas and some special molecules which can adsorb the gas. Each molecule contains two sites at which it can adsorb a single ideal gas particle in either one of two states: one with binding energy \(-\epsilon_0\) and another with \(-\epsilon_0/2\). The box top is a movable piston, which keeps the gas pressure constant. The box is surrounded by a heat bath that keeps the temperature constant. Determine the fraction of the molecules with exactly one adsorption site full, as a function of the gas pressure and temperature.

**SOLUTION:**

The Grand Canonical partition function for a single adsorption site is

\[
Z_1 = 1 + \lambda e^{\epsilon_0/\tau} + \lambda e^{\epsilon_0/2\tau},
\]

(17.4.1)

where \(\lambda = e^{\mu/\tau}\) is the absolute activity.

Define \(x = e^{\epsilon_0/2\tau}\) and let \(P_m^n\) be the probability for \(m\) out of \(n\) sites to be full. Then,

\[
P_1^1 = \frac{\lambda x + \lambda x^2}{1 + \lambda x + \lambda x^2} = \frac{\lambda x(1 + x)}{1 + \lambda x + \lambda x^2}.
\]

(17.4.2)

It follows that

\[
P_1^2 = 2 \times \frac{\lambda x(1 + x)}{(1 + \lambda x + \lambda x^2)^2} = 2\lambda^{-1} x^{-2} \frac{1 + x^{-1}}{(\lambda^{-1} x^{-2} + 1 + x^{-1})^2},
\]

(17.4.3)

where we must recall that there are two ways for exactly one site to be full.

The chemical potential of an ideal gas is \(\mu = \tau \log(n/n_Q)\), where \(n_Q = (M\tau/2\pi\hbar^2)^{3/2}\) and \(n = N/V = p/\tau\). Thus, \(\lambda = e^{\mu/\tau} = p/\tau n_Q\). Define \(p_0 = n_Q e^{-\epsilon_0/\tau}\) so that \(\lambda^{-1} x^{-2} = p_0/p\).

Then,

\[
P_1^2 = \frac{2p_0 p(1 + e^{-\epsilon_0/2\tau})}{[p_0 + p(1 + e^{-\epsilon_0/2\tau})]^2}.
\]

(17.4.4)

This is also the fraction of the molecules with exactly one adsorption site full.
Chapter 18

Ideal Gas

- Fermi-Dirac distribution: $f(\varepsilon) = \left[ e^{(\varepsilon - \mu)/\tau} + 1 \right]^{-1}$.
- Bose-Einstein distribution: $f(\varepsilon) = \left[ e^{(\varepsilon - \mu)/\tau} - 1 \right]^{-1}$.
- Ideal gas free energy: $F = N\tau\log\left(\frac{n}{n_Q}\right) - 1$.
- Internal degrees of freedom:
  - $Z \approx 1 + Z_{\text{int}} e^{-\varepsilon_{\text{ext}} - \mu}/\tau$ (when exponential term is small).
  - $\mu \to \mu - \tau \log Z_{\text{int}}$.
  - $F \to F + F_{\text{int}} = F - N\tau \log Z_{\text{int}}$.
  - $\sigma \to \sigma + \sigma_{\text{int}} = \sigma - (\partial F_{\text{int}}/\partial \tau)_{\nu}$.
- Ideal gas law: $p = n\tau$.
- Isothermal expansion: $dU = 0$ and so $dW = -dQ$, where $dW = -pdV$ is the work done ON the gas and $dQ$ is the heat ABSORBED BY the gas.
- Isentropic (usually adiabatic) expansion: $pV^\gamma$ is constant, where $\gamma = C_p/C_V$.
- Two other common ones: isobaric and isochoric.
18.1 Relation of Pressure and Energy Density

[Kittel & Kroemer 6.7]

(a) Show that the average pressure in a system in thermal contact with a heat reservoir is given by

\[ p = -\sum_s (\partial \epsilon_s / \partial V)_N \frac{e^{-\epsilon_s / \tau}}{Z}, \]  

where the sum is over all states of the system.

(b) Show for a gas of free particles that

\[ \left(\frac{\partial \epsilon_s}{\partial V}\right)_N = -\frac{2}{3} \frac{\epsilon_s}{V}, \]  

as a result of the boundary conditions of the problem. The result holds equally whether \( \epsilon_s \) refers to a state of \( N \) noninteracting particles or to an orbital.

(c) Show that for a gas of free nonrelativistic particles,

\[ p = \frac{2U}{3V}, \]  

where \( U \) is the thermal average energy of the system. This result is not limited to the classical regime; it holds equally for fermions and bosons, as long as they are nonrelativistic.

**SOLUTION:**

(a) This is pretty much by definition. We can call \( p_s = (\partial \epsilon_s / \partial V)_N \) the pressure in state \( s \). If you want, we can say \( Z = \sum_s e^{-\epsilon_s / \tau} \), then \( F = \tau \log Z \), and then \( p = -(\partial F / \partial V)_{\tau,N} \), which gives the same result.

(b) We need to know how \( \epsilon_s \) depends on \( V \). These are the energy states of a free particle in a box of volume \( V \). Recall that the energies go like \( L^{-2} = V^{-2/3} \), from which (18.1.2) naturally follows.

(c) Just plug (18.1.2) into (18.1.1).
18.2 Ideal Gas with Two Internal States

[Fall 2005 Modern (Afternoon), Problem 6; Kittel & Kroemer 6.9] Consider an ideal gas of \( N \) monoatomic particles. The particle’s motion may be considered classical but the individual particles have an internal energy which can have value 0 or \( \epsilon \).

(a) Beginning with the partition function of the two-level particles, calculate the contribution to the heat capacity, \( C_V \), of the two levels.

(b) Beginning with the equipartition theorem, calculate the contribution to the heat capacity of the particles due to their translational motion in 3 dimensions.

(c) Place the particles in a volume \( V \) and at temperature \( \tau \). Find the (i) chemical potential; (ii) free energy; (iii) entropy; (iv) pressure; (v) heat capacity at constant pressure.

**SOLUTION:**

(a) The internal partition function is \( Z_{\text{int}} = 1 + e^{-\epsilon/\tau} \). The average energy of the internal states (for \( N \) particles) is

\[
U_{\text{int}} = \frac{N\epsilon e^{-\epsilon/\tau}}{1 + e^{-\epsilon/\tau}} = \frac{N\epsilon}{e^{\epsilon/\tau} + 1}.
\]

Therefore, the contribution of the internal states to \( C_V \) is

\[
C_{\text{int}} = \frac{\partial U_{\text{int}}}{\partial \tau} = \frac{N(\epsilon/\tau)^2 e^{\epsilon/\tau}}{(e^{\epsilon/\tau} + 1)^2}.
\] (18.2.2)

(b) The particles have three quadratic translational kinetic energy terms, each accompanied by one factor of \( \tau/2 \) of energy. Therefore, just due to translation, \( U_{\text{trans}} = \frac{3}{2} N \tau \), and the contribution to the heat capacity is

\[
C_{V,\text{trans}} = \frac{3N}{2}.
\] (18.2.3)

(c) (i) \( \mu = \tau \log(n/nQZ_{\text{int}}) \).
(ii) \( F = N\tau[\log(n/nQ) - 1] - N\tau \log Z_{\text{int}} = N\tau[\log(n/nQZ_{\text{int}}) - 1] \).
(iii) \( \sigma_{\text{int}} = -\partial F_{\text{int}}/\partial \tau = N \log Z_{\text{int}} + \frac{N\epsilon}{\tau} [e^{\epsilon/\tau} + 1]^{-1} \). We add this to Sackur-Tetrode to get

\[
\sigma = N[\log(nQZ_{\text{int}}/n) + \frac{5}{2} + \frac{\epsilon/\tau}{e^{\epsilon/\tau} + 1}].
\]

(iv) \( F_{\text{int}} \) is \( V \)-independent. Hence, \( p = -\partial F/\partial V \)\( _{\tau,N} = N\tau/V \), as usual.
(v) \( C_P = \frac{5}{2} N + C_{\text{int}} \) with \( C_{\text{int}} \) given in (18.2.2).
18.3 Ideal Gas Pressure and Small Oscillations

[Lim 1019] An ideal gas is contained in a large jar of volume $V_0$. Fitted to the jar is a glass tube of cross-sectional area $A$ in which a metal ball of mass $M$ fits snugly. The equilibrium pressure in the jar is slightly higher than atmospheric pressure, $p_0$, because of the weight of the ball. If the ball is displaced slightly from equilibrium it will execute simple harmonic motion (neglecting friction). If the states of the gas represent a quasistatic adiabatic process and $\gamma$ is the ratio of specific heats, find a relation between the oscillation frequency, $\omega$, and the variables of the problem.

**Figure 18.1: Ideal gas in a jar.**

**SOLUTION:**

Let $p$ and $V$ be the pressure and volume of the air in the jar at any given time. Since the compression and expansion of the air is presumed adiabatic, $pV^\gamma$ is constant. Hence, $V^\gamma \, dp + \gamma pV^{\gamma-1} \, dV = 0$, or $dp = -\frac{\gamma}{V} \, dV$. Note that the restoring force on the mass is $F = A \, dp$ and the small displacement, $x$, of the mass leads to the small variation of the volume via $dV = Ax$. Finally, the nominal equilibrium pressure in the jar is found by balancing forces on the mass: $pA = p_0A + mg$ and so $p = p_0 + mg/A$. Putting this all together gives

$$F = A \, dp = -\frac{\gamma pA \, dV}{V} = -\frac{\gamma (p_0 + mg/A)A(Ax)}{V_0 + Ax} \approx -\frac{\gamma A^2 (p_0 + mg/A)}{V_0} \, x,$$

(18.3.1)

where we have dropped all terms higher than linear in $x$.

Then, the FoSO is given by

$$\omega^2 = \frac{\gamma A^2 (p_0 + mg/A)}{mV_0}.$$  \hspace{1cm} (18.3.2)

**Aside:** You may have noticed that I did not put in a perturbation in $p$ on the RHS when I plugged in $p = p_0 + mg/A$. I could have, but of course, since there is already a $dV$ in the numerator, any such term would necessarily be higher order. In fact, this same argument could have been used to argue that we could have just substituted in $V_0$ for $V$ in the denominator, rather than $V = V_0 + Ax$. 
18.4 Mixing and Compression of Ideal Gas

[Fall 2005 Classical (Morning), Problem 3] A thermally insulated cylinder with an insulating piston at one end has a volume 1 m$^3$. The volume is divided in half by an insulating partition. One side of the partition is filled with 10 moles of N$_2$ gas at 127°C and the other side is filled with 5 moles of CO$_2$ gas at 27°C. The partition is removed and equilibrium established whilst the piston remains immobile. Then the piston compresses the mixture in an adiabatic fashion until it reaches the point where the partition used to be.

(a) Determine the temperature, total pressure, entropy change of each component, and total entropy change of the gas when equilibrium is established after the partition has been removed, but before the piston has moved.

(b) Do the same after the piston has reached the point where the partition used to be.

Note: $C_V = 20.6$ J/(K·mol) for N$_2$ and $C_V = 28.2$ J/(K·mol) for CO$_2$.

Figure 18.2: Fall 2005 Classical (Morning), Problem 3

SOLUTION:

(a) Let subscript 1 and 2 mean N$_2$ and CO$_2$, respectively. For an ideal gas, the energy per mole, $u$, is a function only of $T$. As given to us, $C_V$ in units of J/(K·mol) is defined by $C_V = \frac{\partial u}{\partial T}$. Therefore, up to a constant, $U(T) = nC_V T$.

Since the system is isolated, $Q = 0$, and since there is no volume change, $W = 0$. It follows that $\Delta U = 0$ and so $U_i = U_1(T_1) + U_2(T_2) = U_f = U_1(T_f) + U_2(T_f)$, where we have used the fact that at thermal equilibrium, the temperature is $T_f$ for both gases. Hence,

$$T_f = \frac{n_1C_{V,1}T_1 + n_2C_{V,2}T_2}{n_1C_{V,1} + n_2C_{V,2}} = 360 \text{ K} = 87°C$$  \hspace{1cm} (18.4.1)

Since each gas is independent of the other, we can treat the final state as an ideal gas with $n_1 + n_2$ moles and a final temperature of $T_f$. Hence,

$$P = (n_1 + n_2)RT_f/V = 4.49 \times 10^4 \text{ Pa}.$$  \hspace{1cm} (18.4.2)

Note that 1 Pa is 1 N/m$^2$.

The thermodynamic identity reads $T dS = dU + P dV = nC_V dT + nRT \frac{dV}{V}$. Thus,

$$\Delta S = nC_V \ln \frac{T_f}{T_i} + nR \ln \frac{V_f}{V_i}.$$  \hspace{1cm} (18.4.3)

Plugging in the numbers, we find

$$\Delta S_1 = 35.9 \text{ J/K} \quad \Delta S_2 = 54.5 \text{ J/K} \quad \Delta S_{\text{tot}} = 90.4 \text{ J/K}.$$  \hspace{1cm} (18.4.4)

(b) $\Delta S_1 = \Delta S_2 = \Delta S_{\text{tot}} = 0$ since the process is reversible and the two gases are non-interacting.

Along an adiabat, $PV^\gamma$ and $TV^{\gamma-1}$ are constant. The heat capacity at constant pressure is $C_P = \left(\frac{\partial u}{\partial T}\right)_P + P \left(\frac{\partial v}{\partial T}\right)_P$, where $v = RT/P$ is the molar volume. Hence, $\left(\frac{\partial u}{\partial T}\right)_P = \frac{R}{\gamma}$. On the
other hand, since \( u \) is a function only of \( T \), we have \( \left( \frac{\partial u}{\partial T} \right)_P = \left( \frac{\partial u}{\partial T} \right)_V \). Hence, \( C_P = C_V + R \) and so \( \gamma = \frac{C_P}{C_V} = 1 + \frac{R}{C_V} \).

For two gases,
\[
\gamma = \frac{n_1 C_{P,1} + n_2 C_{P,2}}{n_1 C_{V,1} + n_2 C_{V,2}} = 1.36. \tag{18.4.5}
\]

Hence, \( P_f = P_i \left( \frac{V_i}{V_f} \right)^\gamma = 1.15 \times 10^5 \text{ Pa} \)
and \( T_f = T_i \left( \frac{V_i}{V_f} \right)^{\gamma - 1} = 462 \text{ K} = 189^\circ \text{C} \).
Chapter 19

Fermi and Bose Gases

- Fermi energy: $\epsilon_F = \frac{\hbar^2}{2M} \left( \frac{3\pi^2 N}{\tau} \right)^{2/3}$ in 3D.
- Kinetic energy in ground state: $U_0 = \frac{3}{5} N \epsilon_F$.
- Density of states in 3D: $D(\epsilon) = \frac{3N}{2\epsilon}$.
- Procedure for density of states for free particles: find $N$ as a function of the appropriate desired variable, $x$, and then $D(x) = dN(x)/dx$.
- Heat capacity of an electron gas at $\tau \ll \tau_F$ is $C_{el} = \frac{1}{3} \pi^2 D(\epsilon_F) \tau = \frac{1}{2} \pi^2 N \tau / \epsilon_F$.
- For a Bose gas at $\tau < \tau_E$, the fraction of atoms in excited orbitals is $N_e/N = 2.612 n_Q/n \sim (\tau/\tau_E)^{3/2}$.
- The Einstein condensation temperature of a gas of noninteracting bosons is $\tau_E = \frac{2\pi\hbar^2}{M} \left( \frac{N}{2\pi M^2} \right)^{2/3}$.
19.1 General Free Fermi Gas

[Spring 2008 Modern (Afternoon), Problem 6] Consider a non-interacting Fermi gas of \( N \) particles in a volume \( V \) at temperature \( T = 0 \). Assume that the Fermi energy \( \epsilon_F = A(N/V)^{2/3} \) and the density of states \( D = B\epsilon^{1/2} \), where \( A \) and \( B \) are constants such that \( A^{3/2}B = 3V/2 \). The constant \( A \) does not depend on \( N \) or \( V \); \( B \) does.

(a) Give an expression for the distribution function \( f(\epsilon, T = 0) \).

(b) What is the total ground state kinetic energy \( U \)? Express your answer in a form that is independent of \( A \) and \( B \).

(c) Derive an expression connecting the pressure and volume of the gas. Express your answer in a form that is independent of \( A \) and \( B \).

(d) Find the compressibility \( K = -\left[ V \left( \frac{\partial p}{\partial V} \right) \right]^{-1} \) in terms of \( N \), \( V \) and \( \epsilon_F \).

**SOLUTION:**

(a) Since \( T = 0 \), this is just a step function:

\[
f(\epsilon, T = 0) = \begin{cases} 
1, & \epsilon < \epsilon_F, \\
0, & \epsilon > \epsilon_F.
\end{cases}
\]  

(19.1.1)

(b) First, let see why the condition relating \( A \) and \( B \) is necessary. The integral of the density of states had better count the number of particles correctly, which we find that it indeed does:

\[
\int_0^{\epsilon_F} B\epsilon^{1/2} \, d\epsilon = \frac{2}{3}BA^{3/2}N/V = N.
\]  

(19.1.2)

To calculate the ground state kinetic energy, we just insert another \( \epsilon \):

\[
\begin{align*}
U &= \int_0^{\epsilon_F} B\epsilon^{3/2} \, d\epsilon \\
&= \frac{2}{5}BA^{3/2}(N/V)\epsilon_F = \frac{3}{5}N\epsilon_F.
\end{align*}
\]  

(19.1.3)

(c) \( p = -(\partial U/\partial V)_{\sigma,N} \), where \( U \) here stands for the total internal energy, not necessarily just the ground state energy. However, we are only asked to consider the ground state and in that case, a change of volume while remaining in the ground state constitutes no change in entropy and thus the process is automatically isentropic. We can drop the \( \sigma \) subscript and can safely calculate the pressure:

\[
p = -\left( \frac{\partial U}{\partial V} \right)_N = -3\frac{N}{5}\left( -\frac{2\epsilon_F}{3V} \right) = \frac{2N}{5V}\epsilon_F = \frac{2}{5}n\epsilon_F.
\]  

(19.1.4)

(d) \( K^{-1} = -V\left( \frac{\partial p}{\partial V} \right) = V\frac{1}{V}\frac{2}{5}n\epsilon_F(1 + \frac{2}{3}) = \frac{2}{3}n\epsilon_F \)
19.2 Fermion Degeneracy Pressure

(Spring 2009 Modern (Afternoon), Problem 2) Consider a solid containing \( N \) atoms each of which contributes \( q \) free electrons contained in a volume \( V \) (which can be taken as a cube).

(a) Calculate the Fermi momentum \( k_F = p_F/h \), in terms of the free energy density \( \rho \equiv Nq/V \).

(b) Derive the expression for the total energy of the Fermi gas of electrons,
\[ E_{\text{tot}} = \frac{\hbar^2(3\pi^2Nq)^{5/3}}{10\pi^2m_e} V^{-2/3}. \]  
(19.2.1)

(c) Derive the expression for the quantum pressure \( P \) (degeneracy pressure).

SOLUTION: [Thanks to Aaron]

(a) The number of states up to the wavevector, \( k \), is \( N(k) = 2 \times \frac{1}{8} \times \frac{4}{3}\pi n(k)^3 \), where the
\( k = \frac{\pi}{L} n \) and so \( n(k) = \frac{k}{L} k \). Thus, \( N(k) = \frac{\pi^2}{3} \pi^3 k^3 = \frac{V}{3\pi^2} k^3 \). We want the number of states up to the Fermi wavevector, \( k_F \), to be the total number, \( Nq \), of electrons. Hence, \( Nq = N(k_F) = \frac{V}{3\pi^2} k_F^3 \) and so
\[ k_F = \left(3\pi^2\rho\right)^{1/3} \]  
(19.2.2)

(b) You can calculate the Fermi energy the same way. You should find that it’s the same as simply plugging in \( k_F \) in the definition of \( \epsilon(k) \) and so
\[ \epsilon_F = \frac{\hbar^2 k_F^2}{2m_e} = \frac{\hbar^2(3\pi^2Nq)^{2/3}}{2m_e} V^{-2/3} = \frac{\hbar^2(3\pi^2Nq)^{5/3}}{6Nq\pi^2m_e} V^{-2/3}, \]  
(19.2.3)
where I just multiplied above and below by \( 3\pi^2Nq \) for future convenience.

Check for yourself that this problem satisfies the conditions of Problem 19.1 and so the ground state energy is
\[ E_{\text{tot}} = \frac{3}{5} Nq\epsilon_F = \frac{\hbar^2(3\pi^2Nq)^{5/3}}{10\pi^2m_e} V^{-2/3}. \]  
(19.2.4)

(c) As in Problem 19.1, \( P = \frac{2}{5} \rho \epsilon_F = \frac{\hbar^2}{15\pi^2m_e}(3\pi^2\rho)^{5/3} \) Or just take \( P = -\partial U/\partial V \).
19.3 Neutrino Gas

[Spring 2008 Modern (Afternoon), Problem 2] Consider a gas of $N$ neutrinos at temperature $T$ in a cubic box of volume $V$. Assume that the neutrinos are massless, travel at the speed of light, $c$, and have total spin angular momentum $1/2$. They have only one state ($J_z = -1/2$) and not two.

(a) What kind of statistics describes the gas of neutrinos.

(b) What is the relation between the energy, $\mathcal{E}$, and momentum $p$ of a neutrino?

(c) Show that the number of states for this system per unit energy range is $V\mathcal{E}^2/2\pi^2\hbar^3 c^3$.

(d) Find an expression for the energy of the highest occupied energy level at $T = 0$.

SOLUTION:

(a) Fermi-Dirac statistics, since these are spin-1/2 and so they are fermions.

(b) They are massless, hence $\mathcal{E} = pc$.

(c) Let space be a cube with large side-length $L$ (and volume $V = L^3$). Let the origin be one corner and choose the cube to be the positive octant. Require neutrino wavefunctions to vanish at the boundaries, thus requiring that $L$ be a half or whole integer number of wavelengths in each direction: $\lambda_i = 2L/n_i$ for $n_i \in \mathbb{Z}$ and $i = x, y, z$. Thus, the wavevector is $k_i = 2\pi/\lambda_i = \pi n_i/L$ and the volume of each state in momentum space is $(\pi/L)^3$. We integrate over all momentum space and divide by the volume per state to count the number of states, approximating the integral over a cube by one over a sphere, and dividing by 8 to restrict to the positive octant:

$$N(k) = \frac{1}{8} \frac{1}{(\pi/L)^3} \int k^2 dk \sin \theta_k d\theta_k d\phi_k = \frac{Vk^3}{6\pi^2}. \quad (19.3.1)$$

We would like to express this in terms of energy $\mathcal{E} = hkc$:

$$N(\mathcal{E}) = \frac{V\mathcal{E}^3}{6\pi^2\hbar^3 c^3}. \quad (19.3.2)$$

The density (per unit energy) of states is

$$D(\mathcal{E}) = \frac{dN(\mathcal{E})}{d\mathcal{E}} = \frac{VE^2}{2\pi^2\hbar^3 c^3}. \quad (19.3.3)$$

(d) Since $T = 0$, all energy states below the Fermi level, $\mathcal{E}_F$, will be filled. Hence, we must integrate $D(\mathcal{E})$ from 0 to $\mathcal{E}_F$ and set this equal to $N$. Doing so easily gives $\mathcal{E}_F = (6\pi^2 n)^{1/3}\hbar c$, where $n = N/V$ is the particle volume density.
19.4 Leaky Box of Electrons

[Fall 2008 Modern (Afternoon), Problem 2] A cavity containing a gas of electrons has a small hole, of area $A$, through which electrons can escape. External electrodes are so arranged that, if the potential energy of an electron inside the cavity is taken as zero, then its potential energy outside the cavity is $V > 0$. Thus, an electron will leave the leaky box IF it approaches the small hole with a kinetic energy larger than $V$. Estimate the electrical current carried by the escaping electrons assuming that

(i) a constant number density of electrons is maintained inside the cavity,

(ii) these electrons are in thermal equilibrium at a temperature $\tau$ and chemical potential $\mu$,

(iii) interactions between the electrons can be neglected, and

(iv) $V - \mu > \tau$.

SOLUTION:

$$dN = \frac{2}{h^3} d^3r d^3p \left[ e^{(\epsilon - \mu)/\tau} + 1 \right]^{-1},$$

where the overall factor of 2 is due to spin, and where $\epsilon = p^2/2m$. It follows that

$$n(p) \equiv \frac{dN}{dV} = \frac{2}{h^3} \frac{d^3p}{e^{(\epsilon - \mu)/\tau} + 1}.$$  \hspace{1cm} (19.4.1)

Let $\theta$ be the approach angle of an electron towards the hole, which goes between $\theta = 0$ and $\theta = \pi/2$. Then the electron sees an effective area of $A \cos \theta$. The differential rate at which electrons leave the box is

$$dR = (\text{app. area})(\text{density})(\text{speed})$$

$$= \left( A \cos \theta \right) \left( \frac{2}{h^3} \frac{p^2 dp \sin \theta d\theta d\phi}{e^{(\epsilon - \mu)/\tau} + 1} \right) \left( \frac{p}{m} \right).$$  \hspace{1cm} (19.4.2)

The current is $I = -e \int dR$ or

$$I = -\frac{2eA}{m^2} \int_0^{2\pi} d\phi \int_0^{\pi/2} \sin \theta \cos \theta d\theta \int_{\sqrt{2mv}}^{\infty} \frac{p^3 dp}{e^{(\epsilon - \mu)/\tau} + 1}$$

$$\approx -\frac{2\pi eA}{m^2} \int_0^{\infty} p^3 e^{-(\epsilon - \mu)/\tau} dp$$

$$= -\frac{4\pi eA m^2}{h^3} \int_0^{\infty} e^{-(\epsilon - \mu)/\tau} d\epsilon$$

$$= -\frac{4\pi eA m^2}{h^3} \int_0^{\infty} \left( x + \frac{\mu}{\tau} \right) e^{-x} dx,$$  \hspace{1cm} (19.4.3)

where the second line follows from $V - \mu >> \tau$, we have change to the variable $x \equiv (\epsilon - \mu)/\tau$, and where $a \equiv \frac{V - \mu}{\tau} >> 1$. The integral evaluates to

$$\int_a^{\infty} \left( x + \frac{\mu}{\tau} \right) e^{-x} dx = \left( 1 + \frac{\mu}{\tau} \right) e^{-a} = \left( 1 + \frac{V}{\tau} \right) e^{-a}.$$  \hspace{1cm} (19.4.4)

Therefore, the current is

$$I \approx -\frac{4\pi eA m^2}{h^3} \left( 1 + \frac{V}{\tau} \right) e^{-(V - \mu)/\tau}.$$  \hspace{1cm} (19.4.5)
19.5 Degenerate Boson Gas

[Kitel & Kroemer 7.8] Find expressions as a function of temperature in the region $\tau < \tau_E$ for the energy, heat capacity, and entropy of a gas of $N$ noninteracting bosons of spin zero confined to a volume $V$. Put the definite integral in dimensionless form; it need not be evaluated.

**SOLUTION:**

As usual, we set the energy of the ground state equal to zero. Then, we must integrate the energy of the excited states. We will need the appropriate density of states. Assuming the gas to be nonrelativistic and confined to a cube of volume $V = L^3$, occupying the positive octant of coordinates, we require the state wavefunctions to vanish at the boundary, which gives the usual constraint on the wavelengths: $\lambda_j = 2L/n_j$ for $j = x, y, z$. Since the wavevector is $k = 2\pi/\lambda$, we have $k_j = (\pi/L)n_j$ and the energy is $\epsilon = \hbar^2 k^2/2m = \hbar^2 (\pi n/L)^2$, where $n^2 = n_x^2 + n_y^2 + n_z^2$. Let $N(\epsilon)$ be the number of states below energy $\epsilon$ (note: this point has nothing to do with whether or not these states are actually filled; we’re just counting states here.) Then,

$$N(\epsilon) = 1 \times 1 \times \frac{4\pi}{3} n^3 = \frac{V}{6\pi^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \epsilon^{3/2}, \quad (19.5.1)$$

where, since these particles have spin 0, we lose a factor of 2 relative to the Fermi case.

The density of states is thus

$$D(\epsilon) = \frac{dN(\epsilon)}{d\epsilon} = \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \epsilon^{1/2}. \quad (19.5.2)$$

The energy is given by

$$U = \int_0^\infty \epsilon D(\epsilon) f(\epsilon) d\epsilon = \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^\infty \frac{\epsilon^{3/2} d\epsilon}{e^{(\epsilon+\mu)/\tau} - 1}. \quad (19.5.3)$$

In the degenerate case, the chemical potential is very much closer to the ground state energy than are any of the excited states. Hence, we may set $\mu = 0$ for calculating the number of excited state particles or their energy (but not to calculate the number of ground state particles). Define $x = \epsilon/\tau$, then

$$U = \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \frac{3\zeta(5/2)}{16} \left(\frac{2m}{\pi\hbar^2}\right)^{3/2} \frac{\tau^{5/2}}{V} = \frac{3\zeta(5/2)}{2\zeta(3/2)} N(\tau/\tau_E)^{3/2}. \quad (19.5.4)$$

The heat capacity at constant volume is

$$C_V = \left(\frac{dU}{d\tau}\right)_{V,N} = \frac{15}{4} \frac{\zeta(5/2)}{\zeta(3/2)} N(\tau/\tau_E)^{3/2}. \quad \text{The heat capacity is also given by} \ C_V = \tau \left(\frac{\partial U}{\partial \tau}\right)_{V,N}, \text{from which we get} \ \sigma = \frac{5}{2} \frac{\zeta(5/2)}{\zeta(3/2)} N(\tau/\tau_E)^{3/2}. \quad (19.5.5)$$
19.6 Bose Gas in Harmonic Potential

[Kevin G.] This problem concerns the possible effects of external potentials on the condensation of a spinless Bose gas.

(a) Consider a free non-relativistic spinless Bose gas confined in a $d$-dimensional hypercube of side-length $L$. Compute the density of states as a function of momentum and as a function of energy. You may use the surface area of the unit $(d-1)$-sphere, $S^{d-1} \subset \mathbb{R}^d$, given by $2\pi^{d/2}/\Gamma(d/2)$, and its volume is the area divided by $d$.

(b) Calculate the Bose-Einstein condensation temperature for arbitrary $d$ and comment on cases, if any exist, when condensation does not occur. You may use the integral $\int_0^\infty \frac{x^n dx}{e^x - 1} = \Gamma(n+1)\zeta(n+1)$.

(c) Place the gas in an external isotropic harmonic potential of characteristic frequency, $\omega$. Will this tend to facilitate or frustrate condensation, and why? Now, calculate the new condensation temperature for arbitrary $d$. Assume that $L$ is much larger than the range of the potential.

SOLUTION:

(a) Since the particles are free, their wavefunctions may be expressed as linear combinations of plane waves. The boundary conditions (that the wavefunctions vanish at the boundaries of the hypercube) imply that $k_j = \frac{x}{L} n_j$ where $j = 1, 2, \ldots, d$ denotes the $d$ directions, and $n_j = 1, 2, \ldots$. The number of states below a certain value of $n = \sqrt{n_1^2 + \cdots + n_d^2}$ is about $N(n) = 1 \times 2^{-d} \times \frac{2\pi^{d/2}}{\Gamma(d/2)} n^d$. The first factor, 1, counts the number of spin states, the second factor restricts to positive $n_j$ values, and the third is the volume of a $(d-1)$-sphere of radius $n$. If we substitute in $n = \frac{1}{L} k$, where $k = \sqrt{k_1^2 + \cdots + k_d^2}$, then we get $N(k) = \frac{2\pi^{d/2}}{\Gamma(d/2)} k^d$. Now, let us substitute the momentum, $p = h k$, to get $N(p) = \frac{2\pi^{d/2}}{h^d \Gamma(d/2)} p^d$, and then the energy $\epsilon = p^2 / 2m$, to get $N(\epsilon) = \frac{2L^d m^{d/2}}{(2\pi)^{d/2} h^d \Gamma(d/2)} \epsilon^{d/2}$. The density of states is the derivative (we have written $V = L^d$):

\[
D(\epsilon) = \frac{dN(\epsilon)}{d\epsilon} = \left(\frac{m}{2\pi\hbar^2}\right)^{d/2} \frac{V}{\Gamma(d/2)} \epsilon^{d/2-1}. \tag{19.6.1}
\]

(b) The Bose-Einstein condensation temperature is when $N$, the number of particles, is equal to the integral of the density of states from $\epsilon = 0$ to $\epsilon = \infty$, when $\mu \approx 0$, which counts the number of excited particles, $N_e$. Below this temperature, $N_e$ decreases and $N_0$ increases markedly. Thus, setting $\mu = 0$, we have

\[
N_e = \int_{0^+}^\infty d\epsilon D(\epsilon) \frac{1}{e^{\beta\epsilon} - 1} = \left(\frac{m}{2\pi\hbar^2}\right)^{d/2} \frac{V}{\Gamma(d/2)} \int_{0^+}^\infty \frac{\epsilon^{d/2-1} d\epsilon}{e^{\beta\epsilon} - 1} = \left(\frac{m\tau}{2\pi\hbar^2}\right)^{d/2} \zeta(d/2) V. \tag{19.6.2}
\]
Setting this equal to \( N \) gives

\[
\tau_E = \frac{2\pi h^2}{m} \left( \frac{N}{\zeta(d/2) V} \right)^{2/d} \tag{19.6.3}
\]

Compare this with the \( d = 3 \) result given in Kittel Eqn. 72 p.205. Indeed, you can check for yourself that \( \zeta(3/2) = 2.612 \).

Sensible values of \( d \) are \( d = 1, 2, \ldots \). It happens that \( \zeta(1) \) is infinite and \( \zeta(1/2) \) is negative! The \( d = 2 \) case is easy to understand: \( \tau_E = 0 \) in this case, so condensation never occurs. For the \( d = 1 \) case, you can just check that the integral in (19.6.2) just doesn’t converge, which is also the case for \( d = 2 \). Thus, the excited states can accommodate an arbitrarily large population and thus \( N_0 \) will always be negligible and condensation never occurs. For \( d \geq 3 \), condensation occurs at \( \tau_E \) given above.

I will now compute \( N_e \) in a slightly different manner, which will be useful for the later parts of this problem. The hypervolume per quantum state in position-momentum phase space is \( h^d \). Thus, \( N_e \) is given by

\[
N_e = \frac{1}{h^d} \int d^d x \int d^d p \frac{1}{e^{p^2/2m} - 1} = \frac{V}{(2\pi h)^d} \times \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_{0+}^{\infty} \frac{p^{d-1} dp}{e^{\beta p^2/2m} - 1} = \frac{2V}{2^{d-1} \Gamma(d/2)} \frac{1}{2} \left( \frac{2m}{\beta} \right)^{d/2} \int_{0+}^{\infty} \frac{x^{d-1} dx}{e^x - 1} = \left( \frac{m \tau}{2 \pi \hbar^2} \right)^{d/2} \zeta(d/2) V. \tag{19.6.4}
\]

Again, setting this equal to \( N \) and solving for \( \tau \) yields (19.6.3).

(c) The harmonic potential will tend to trap the particles at the equilibrium position and will thus help condensation along.

**Method 1:** Define the six-vector \( \mathbf{a} = \frac{1}{\sqrt{2m\tau}} (m\omega \mathbf{r}, \mathbf{p}) \) so that \( d^d r \, d^d p = (2\tau/\omega)^d d^d a \):

\[
N_e = \frac{1}{h^d} \int d^d x \int d^d p \left[ \exp\left( \frac{m\omega^2 r^2}{2} + \frac{p^2}{2m} \right) - 1 \right]^{-1} = \left( \frac{\tau}{\pi \hbar \omega} \right)^d \frac{2\pi^d}{\Gamma(d)} \int_{0+}^{\infty} \frac{a^{2d-1} da}{e^{ax} - 1} = \left( \frac{\tau}{\hbar \omega} \right)^d \frac{1}{\Gamma(d)} \int_{0+}^{\infty} \frac{x^{d-1} dx}{e^x - 1} = \left( \frac{\tau}{\hbar \omega} \right)^d \zeta(d). \tag{19.6.5}
\]

Setting this equal to \( N \) and solving for \( \tau \) yields

\[
\tau_E = \hbar \omega \left( \frac{N}{\zeta(d)} \right)^{1/d} \tag{19.6.6}
\]
Now, this $\tau_E$ exists and is finite for $d \geq 2$, and thus, in this regime, condensation occurs. But $\tau_E = 0$ when $d = 1$, thus condensation still does not occur for $d = 1$.

**Method 2:** The states of the harmonic potential are labelled by $d$ non-negative integers $n_1, \ldots, n_d$ and the energy is $\epsilon = \hbar \omega (n_1 + \cdots + n_d + \frac{d}{2})$. Define $n = n_1 + \cdots + n_d$, then the $n$th eigenenergy has degeneracy $\binom{n + d - 1}{n}$, which is precisely the number of ways of writing a non-negative integer as the sum of $d$ non-negative integers (c.f. Problem 14.2). By definition,

$$N_e = \sum_{n=1}^{\infty} \binom{n + d - 1}{n} \left[ \exp \beta \left( \hbar \omega (n + \frac{d}{2}) - \mu \right) - 1 \right]^{-1}.$$

(19.6.7)

We just need to be a little careful about $\mu$. Usually, we set $\mu = 0$, but this is really because, as $\tau \rightarrow \tau_E$, the chemical potential approaches the ground state energy from below and gets much closer to it than is the first excited state. In our case, the ground state energy has not been set to zero, but is instead $\epsilon_0 = \hbar \omega n/2$. Thus, we must set $\mu = \hbar \omega n/2$:

$$N_e = \frac{1}{(d - 1)!} \sum_{n=1}^{\infty} \frac{(n + d - 1)!}{n!} \frac{1}{e^{\beta \hbar \omega n} - 1}.$$

(19.6.8)

Write $(d - 1)! = \Gamma(d)$ and approximate $\frac{(n + d - 1)!}{n!} \approx n^{d-1}$, for large $n$. Finally, approximate the sum by an integral:

$$N_e = \frac{1}{\Gamma(d)} \int_{0+}^{\infty} \frac{n^{d-1} \, dn}{e^{\beta \hbar \omega n} - 1} = \left( \frac{\tau}{\hbar \omega} \right)^d \frac{1}{\Gamma(d)} \int_{0+}^{\infty} \frac{x^{d-1} \, dx}{e^x - 1} = \left( \frac{\tau}{\hbar \omega} \right)^d \zeta(d).$$

(19.6.9)

This is the same result as in Method 1 and thus gives the same $\tau_E$. 
Energy conservation: \( dU = dW + dQ \), where \( dW \) is the work done ON the gas and \( dQ \) is the heat ABSORBED by the gas.

For a reversible process: \( dQ = \tau \, d\sigma \).

Carnot efficiency (heat engine): \( \eta_C = 1 - \frac{\tau_l}{\tau_h} \).

Carnot refrigerator COP: \( \gamma_C^{-1} = \left( \frac{\tau_h}{\tau_l} \right) - 1 \).
20.1 A Dam

[Lim 1037] In the water behind a high power dam (110 m high), the temperature difference between the surface and bottom may be 10°C. Compare the possible energy extraction from the thermal energy of a gram of water with that generated by allowing the water to flow over the dam through turbines in the conventional manner. [Assume that the specific heat of water is constant over the temperature range and is $c_V = 4.15 \text{ J/g} \cdot \text{K}$. The top surface of the water will be warmer and take its temperature to be room temperature]

**SOLUTION:**

Assume that the engine extracting the energy from the water at the high and low temperatures is ideal (Carnot, say). Its efficiency is taken to be $\eta = 1 - \frac{T_l}{T_h}$. The energy extracted is

$$W = \eta Q = (1 - \frac{T_l}{T_h}) mc_\text{p} (T_h - T_l) \approx 1.42 \text{ J}.$$  \hspace{1cm} (20.1.1)

Again, assuming perfect efficiency of the turbines, the work retrieved from the falling water is $W' = mgh$.

$$W' = mgh = 1.08 \text{ J}.$$  \hspace{1cm} (20.1.2)

They’re close, but $W' < W$. That is, all things being ideal, we would retrieve more energy just by using the temperature gradient in the water to run a heat engine. It presumably is the case that, in reality, the latter procedure is less efficient or otherwise less practical.
20.2 Maximum Work

[Kevin G.] An arbitrary heat engine operates between two reservoirs, each of which has the same temperature-independent heat capacity, $C$. The reservoirs have initial temperatures $\tau_l$ and $\tau_h$, and the engine operates until the reservoirs reach the same final temperature. What is the maximum amount of work obtainable from the engine.

**SOLUTION:**

The change in entropy of the system is

$$\Delta \sigma = \left( \int_{\tau_l}^{\tau_f} + \int_{\tau_h}^{\tau_f} \right) \frac{C \, d\tau}{\tau} = C \ln \frac{\tau_f^2}{\tau_l \tau_h}, \quad (20.2.1)$$

where we can take $C$ out of the integral since it is $\tau$-independent.

By the second law, $\Delta \sigma \geq 0$, which implies

$$\tau_f \geq \sqrt{\tau_l \tau_h}. \quad (20.2.2)$$

For maximum work, the process must be reversible, or $\Delta \sigma = 0$, or $\tau_f = \sqrt{\tau_l \tau_h}$. This is also clear if we just write the work, $W = Q_h - Q_l$:

$$W = C(\tau_h - \tau_f) - C(\tau_f - \tau_l) = C(\tau_h + \tau_l - 2\tau_f) \leq C(\sqrt{\tau_h} - \sqrt{\tau_l})^2. \quad (20.2.3)$$
20.3 The Diesel Cycle

[Kevin] The diesel cycle goes: (1) adiabatic compression; (2) isobaric expansion; (3) adiabatic expansion; (4) isochoric cooling. Let \((p_i, V_i, \tau_i)\) be the pressure, volume and temperature at the start of each step \(i = 1, 2, 3, 4\). The compression ratio is defined to be \(r = V_1/V_2\) and the cut-off ratio is defined to be \(\alpha = V_3/V_2\). Let \(\gamma = C_p/C_V\) be the ratio of heat capacities for the working substance, assumed to be an ideal gas. Calculate the efficiency of this cycle in terms of \(r, \alpha\) and \(\gamma\).

SOLUTION:

Let us compute the thermodynamic variables at each stage in terms of \((p_1, V_1)\). Let \(N\) be the fiducial number of molecules of the working substance. Then, \(\tau_1 = p_1 V_1/N\). Along the adiabatic compression curve, \(pV^\gamma\) is constant and, by definition, \(V_2 = V_1/r\). Hence, \(p_2 = p_1(V_1/V_2)^\gamma = r^\gamma p_1\) and \(\tau_2 = r^{\gamma - 1} p_1 V_1/N\). At step 3, \(p_3 = p_2 = r^\gamma p_1\) and \(V_3 = \alpha V_2 = \alpha V_1/r\); thus, \(\tau_3 = r^{\gamma - 1} \alpha p_1 V_1/N\). After the adiabatic expansion, \(V_4 = V_1\), and \(p_4 = \alpha^\gamma p_1\) and \(\tau_4 = \alpha^\gamma p_1 V_1/N\). In summary,

\[
\begin{align*}
\tau_1 &= p_1 V_1/N, \\
(p_2, V_2, \tau_2) &= (r^\gamma p_1, V_1/r, r^{\gamma - 1} p_1 V_1/N), \\
(p_3, V_3, \tau_3) &= (r^\gamma p_1, \alpha V_1/r, r^{\gamma - 1} \alpha p_1 V_1/N), \\
(p_4, V_4, \tau_4) &= (\alpha^\gamma p_1, V_1, \alpha^\gamma p_1 V_1/N).
\end{align*}
\]

Now, let us compute the heat. There is no heat for the adiabats. The heat is absorbed by the gas for \(2 \rightarrow 3\) and released by the gas for \(4 \rightarrow 1\). The heat absorbed is \(Q_{2\rightarrow3} = NC_p(\tau_3 - \tau_2) = C_p r^{\gamma - 1}(\alpha - 1)p_1 V_1\). The heat released is \(Q_{4\rightarrow1} = NC_V(\tau_1 - \tau_4) = -C_V(\alpha^\gamma - 1)p_1 V_1\). We have \(Q_{in} = Q_{2\rightarrow3}\) and \(Q_{out} = |Q_{4\rightarrow1}|\). Hence, the efficiency is

\[
\eta = \frac{Q_{in} - Q_{out}}{Q_{in}} = 1 - \frac{C_V(\alpha^\gamma - 1)p_1 V_1}{C_p r^{\gamma - 1}(\alpha - 1)p_1 V_1} = 1 - \frac{1}{\gamma r^{\gamma - 1} \left(\frac{\alpha^\gamma - 1}{\alpha - 1}\right)}.
\]
20.4 Heat, Work, Energy and Entropy

[Fall 2008 Classical (Afternoon), Problem 1]

(a) A gas consisting of 2.0 moles of diatomic nitrogen (N₂) is initially at a temperature of 27.0°C, and occupies a volume of 0.020 m³. The gas first expands at a constant pressure until the volume has doubled, and then adiabatically until the temperature returns to its initial value.

(i) Draw a diagram of the process in the P-V plane.

(ii) What is the total change in the internal energy of the nitrogen?

(iii) What is the total work done by the nitrogen?

(iv) What is the total heat supplied in the process?

(v) What is the final volume?

(b) (i) Two 1200-kg cars are travelling at 80 km/h in opposite directions when they collide and are brought to rest. Calculate the resulting change in entropy of the Universe. Assume T = 20°C.

(ii) A 130-g insulated Al cup at 20°C is filled with 210 g of water at 90°C. Calculate the total change in entropy of this system after reaching equilibrium. (The specific heat of Al is 900 J/kg·°C, and for water is 4186 J/kg·°C.)

**SOLUTION:** [Thanks to James]

(a) (i) Let P₀, V₀, T₀ be the initial pressure, volume and temperature, P₁ = P₀, V₁ = 2V₀, T₁ after the first step, and P₂, V₂, T₂ = T₀ after the second step.

Assume N₂ is ideal, then P₀ = nRT/V = 250 kPa. For a diatomic gas at not too extreme a temperature, the number of quadratic degrees of freedom is f = 5 (three translational and two rotational). Quantum mechanics prohibits rotation along the axis connecting the two nitrogen atoms since that is an axis of continuous symmetry. By equipartition, \( U = \frac{5}{2} N k T = \frac{5}{2} n R T \).

Heat capacity is defined as the amount of heat required to change the temperature of a material: \( C = \frac{d Q}{dT} \). By the first law, \( C = \frac{d U - d W}{dT} = \frac{d U + P d V}{dT} \). Thus, the heat capacity at constant volume is \( C_V = \frac{\partial U}{\partial T}\bigg|_V = \frac{5nR}{V} \) and an constant pressure is \( C_P = \frac{\partial U}{\partial T}\bigg|_P = P\frac{\partial V}{\partial T}\bigg|_P = 5nR/2 + P(nR/P) = 7nR/2 \). The adiabat constant is \( \gamma \equiv C_P/C_V = 7/5 \) and, along an adiabat, the relation \( PV^{\gamma} \) = constant holds.

We have \( T_2 = T_0 \) or \( P_2 V_2 = P_0 V_0 \). But, we also have \( P_2 V_2^{\gamma} = P_1 V_1^{\gamma} = 2^{\gamma} P_0 V_0^{\gamma} \).

We can divide these two equations to solve for \( V_2 \) and then solve for \( P_2 \). We get \( V_2 = 2^{\gamma/(\gamma - 1)} V_0 = 0.226 \text{ m}^3 \) and \( P_2 = 2^{-\gamma/(\gamma - 1)} P_0 = 22.1 \text{ kPa} \).

Now, we can draw the P-V diagram with accurate endpoints.

(ii) The total change in internal energy is \( \Delta U = 0 \) since \( \Delta T = 0 \).

(iii) The work done by the gas is the area under the graph. The work during step 1 is \( W_{0 \rightarrow 1} = P_0(V_1 - V_0) = P_0 V_0 \). Along the adiabat, \( PV^{\gamma} = P_2 V_2^{\gamma} = 2^{\gamma} P_0 V_0^{\gamma} \):

\[
W_{1 \rightarrow 2} = \int P \, dV = 2^{\gamma} P_0 V_0^{\gamma} \int_{V_1}^{V_2} V^{-\gamma} \, dV = 2^{\gamma} P_0 V_0^{\gamma} \left( \frac{1}{\gamma - 1} \left( \frac{1}{V_1^{\gamma - 1}} \right) - \frac{1}{V_2^{\gamma - 1}} \right). \tag{20.4.1}
\]

After some algebra, this can be simplified to

\[
W_{1 \rightarrow 2} = \frac{1}{\gamma - 1} P_0 V_0. \tag{20.4.2}
\]
Thus, the total work done by the gas is

\[ W_{0\rightarrow 2} = W_{0\rightarrow 1} + W_{1\rightarrow 2} = \frac{\gamma}{\gamma-1} P_0 V_0 = \frac{\gamma}{2} P_0 V_0 = 17.4 \text{ kJ}. \] (20.4.3)

(iv) Since \( \Delta U = 0 \), we have \( Q = W = 17.4 \text{ kJ} \).

(v) \( V_2 = 0.226 \text{ m}^3 \).

(b) (i) In Kelvin, the temperature is \( T = 293.15 \text{ K} \). Assume the process is isothermal (surely this is a terrible assumption!) Then, \( \Delta S = \Delta U/T \) and \( \Delta U \) is just the total initial kinetic energy, which is converted to other forms of energy during the crash. Then, \( \Delta U = 2\left(\frac{1}{2}mv^2\right) = 593 \text{ kJ} \) and so \( \Delta S = 2.02 \text{ kJ/K} \).

(ii) To compute the final temperature, \( m_w c_w (T_f - T_w) + m_{Al} c_{Al} (T - T_{Al}) = 0 \):

\[ T_f = \frac{m_w c_w T_w + m_{Al} c_{Al} T_{Al}}{m_w c_w + m_{Al} c_{Al}} = 81.8^\circ. \] (20.4.4)

The change in entropy is, after changing the temperatures to Kelvin,

\[ \Delta S = m_{Al} c_{Al} \int_{T_{Al}}^{T_f} \frac{dT}{T} + m_w c_w \int_{T_w}^{T_f} \frac{dT}{T} = 2.24 \text{ J/K}. \] (20.4.5)
20.5 Geothermal Engine

[Fall 2005 Modern (Afternoon), Problem 4] To help the “energy crisis” a group of physicists decide to use a geothermal area to operate a heat engine to produce electricity. They discover a 30 km$^3$ region of hot rock underground with a temperature of $T_o = 600^\circ C$. Water is pumped into the rock and the emerging steam used to run the electric generator. This may be thought of as a heat engine whose ambient exhaust temperature is the atmosphere ($T_a = 20^\circ C$). As the rock cools the rate of steam production decreases and the physicists plan to quit the project when the rock temperature has dropped to $T_f = 110^\circ C$. Find the maximum amount of electrical energy (in kWhr) that can be generated. The rock’s density is $\rho = 7000$ kg/m$^3$ and the rock’s specific heat is $C = 10^3$ J/kg·K.

SOLUTION:

A Carnot engine will maximize the work out. The efficiency is $\eta = 1 - \frac{T_a}{T}$, with $T$ the rock temp. The heat drawn out of the rock is $dQ_H = -mcdT = -\rho Vc dT$. The work out is thus $dW = \eta dQ_H = -\rho Vc(1 - \frac{T_f}{T}) dT$. The total work is

$$W = -\rho Vc \int_{T_o}^{T_f} \left(1 - \frac{T_f}{T} \right) dT = \rho Vc T_a \left[ x_o - x_f - \ln(x_o/x_f) \right], \quad (20.5.1)$$

where $x_o = T_o/T_a$ and $x_f = T_f/T_a$.

Plugging in numbers, $x_o = \frac{600 + 273.15}{20 + 273.15} = 2.979$ and $x_f = \frac{110 + 273.15}{20 + 273.15} = 1.307$. Remember that there are $3.6 \times 10^3$ J in a kWhr. The final answer is $W = 1.45 \times 10^{13}$ kWhr.
20.6 Modified Carnot Cycle

[Fall 2009 Modern (Morning), Problem 5] Consider the following engine running in a modified Carnot cycle described by the figure below. Here, $T_2 = 2T_1$, $T_3 = 4T_1$, $\sigma_2 = 2\sigma_1$ and $\sigma_3 = 3\sigma_1$.

(a) During which parts of the cycle is heat absorbed by the working substance? During which parts is heat transferred to the outside world?

(b) Calculate the work done on the outside world by the engine.

(c) Calculate the efficiency of the engine.

(d) How does the efficiency calculated in part (c) compare to the efficiency of a standard Carnot engine operating at temperatures $T_3$ and $T_1$?

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure20.3}
\caption{Figure 20.3: Fall 2009 Modern (Morning), Problem 5}
\end{figure}

SOLUTION:

(a) $dQ = T \, d\sigma$, so $Q = 0$ for steps 1, 3 and 5. Heat is adsorbed ($dQ, d\sigma > 0$) during step 2 and heat is transferred to the environment ($dQ, d\sigma < 0$) during steps 4 and 6.

(b) $\Delta U = 0$ for a whole cycle; so, $W = Q$. The total $A$ is just the area of the shape: $Q = (2\sigma_1)(3T_1) - \sigma_1T_1 = 5\sigma_1T_1$. So, $W = 5\sigma_1T_1$.

(c) $Q_{\text{abs}} = Q_2 = T_3(\sigma_3 - \sigma_1) = 8\sigma_1T_1$, so $\eta = W/Q_{\text{abs}} = 5/8$.

(d) Standard Carnot: $\eta_C = 3/4$. We can calculate this by dividing the area of the big rectangle (not gnomon), which is $6\sigma_1T_1$, by the $Q_2 = 8\sigma_1T_1$. Or, $\eta_C = 1 - \frac{T_1}{T_3} = 1 - \frac{1}{4} = \frac{3}{4}$. Thus, $\eta = \frac{5}{6} \eta_C$. 

Chapter 21

Gibbs Free Energy and Phase Transformations

- $G = U - \tau \sigma + pV = F + pV$ is a minimum at thermal equilibrium at constant $\tau$ and $p$.
- $G(\tau, pN) = N\mu(\tau, p)$.
- Clausius-Clapeyron: $\frac{dp}{dT} = \frac{\mu_s - \mu_v}{\nu_s - \nu_v} = \frac{L}{T\Delta\mu}$, where the small letters mean “per mole” and $L$ is the latent heat of vaporization per mole.
21.1 Gibbs and Clausius-Clapeyron

[Fall 2007 Modern (Morning), Problem 1] Consider a cylinder containing some liquid and some gas of a single substance. The movable piston above has a constant force applied, to hold the contents of the cylinder at the constant pressure \( P_0 \). The cylinder is also in contact with a heat bath at temperature \( T_0 \). The system is in thermal and mechanical equilibrium.

(a) Now suppose a small number of molecules \( \Delta N \) of the substance pass from the liquid to the gas phase. Some heat \( \Delta Q \) flows in from the heat bath to maintain the temperature \( T_0 \), and the piston moves slightly, changing the volume by \( \Delta V \) to maintain pressure \( P_0 \). Use thermodynamic arguments to explain why \( \Delta G = 0 \), where \( G = U + PV - TS \) is the Gibbs free energy of the system. State the physical principles clearly. Let \( g_l \) and \( g_g \) be the Gibbs free energies per unit mole of the liquid or gas, respectively. Derive a relationship between \( g_l \) and \( g_g \). If you cannot derive this relationship but you remember what it is, you will receive some credit just for writing it down.

(b) Now suppose the piston is clamped and the heat bath removed. Another source of heat is applied to the cylinder, and a small amount of heat \( \Delta Q \) enters the system. As a result, there are small changes in the temperature and pressure, \( \Delta T \) and \( \Delta P \), and some amount of liquid vaporizes. Express \( \Delta P/\Delta T \) in terms of \( s_l \) and \( s_g \) and \( v_l \) and \( v_g \), which are the entropy and volume per unit mole of the liquid and gas. This is the Clausius-Clapeyron equation. Again, you will receive some credit just for quoting this equation, without deriving it.

(c) A sample of ammonia gas is placed in a cylinder of fixed volume at room temperature. The cylinder is cooled to 220K, whereupon some liquid appears in the cylinder, and the pressure is measured to be 262 mm of Hg. As the temperature is lowered, more liquid ammonia forms (more gas converts to liquid); it is found that 25,400 J of heat must be removed to liquefy one mole of ammonia. The cylinder is then cooled to 180K, at which point the cylinder contains some solid ammonia but no liquid, and the pressure is measured to be 8.8 mm of Hg. As the temperature is lowered it is found that more gas converts to solid, and that 31,200 J must be removed to convert one mole of gas to solid. Find the temperature and pressure at the triple point of ammonia. You may take the approximation that the vapor is an ideal gas, and that it has molar volume much larger than either the solid or the liquid.

SOLUTION:

(a) Quasistatic and reversible implies \( Q = T\Delta S \) and since it is isothermal and isobaric, \( \Delta G = V\Delta P - S\Delta T + \mu\Delta N = \mu\Delta N \). For the whole system, \( \Delta N = \Delta N_g + \Delta N_l = 0 \), and so \( \Delta G = 0 \). We have assumed \( \mu_l = \mu_g \), which must be true if the two phases coexist in equilibrium.

The total Gibbs free energy is \( G = g_g N_g + g_l N_l \) and so \( 0 = \Delta G = g_g \Delta N_g + g_l \Delta N_l = (g_g - g_l)\Delta N_g \), where the last step follows from \( \Delta g_l = -\Delta g_g \). Thus, \( g_l = g_g \).

(b) The chemical potentials of the gas and liquid phase will now change since the pressure and the temperature change. To first order,

\[
\begin{align*}
\mu_g &\rightarrow \mu_g + \left( \frac{\partial \mu_g}{\partial P} \right)_T dP + \left( \frac{\partial \mu_g}{\partial T} \right)_P dT, \\
\mu_l &\rightarrow \mu_l + \left( \frac{\partial \mu_l}{\partial P} \right)_T dP + \left( \frac{\partial \mu_l}{\partial T} \right)_P dT.
\end{align*}
\]

(20.1.1a) (20.1.1b)

However, if the phases are to remain in coexistence at equilibrium, the chemical potentials must remain equal, which implies

\[
\frac{dP}{dT} = \left( \frac{\partial \mu_a}{\partial P} \right)_T - \left( \frac{\partial \mu_l}{\partial P} \right)_T.
\]

(21.1.2)
We can get the entropy from $G$ via $S = -(\partial G / \partial T)_{P,N}$. However, since $G = \mu N$, we also have $(\partial G / \partial T)_{P,N} = N(\partial \mu / \partial T)_P$, from which it follows that $s = -N_A(\partial \mu / \partial T)_P$, where $s = S/n$ and $n = N/N_A$ is the number of moles. Likewise, from $V = (\partial G / \partial P)_{T,N}$, we find $v = N_A(\partial \mu / \partial P)_T$. Hence,

\[
\frac{dP}{dT} = \frac{s_g - s_l}{v_g - v_l}
\]  

(21.1.3)

(c) We can write Clausius-Clapeyron as

\[
\frac{dP}{dT} \approx \frac{T(s_g - s_l)}{Tv_g} = \frac{L}{Tv_g} = \frac{LP}{RT^2},
\]

(21.1.4)

where $L$ is the latent heat per mole and, similarly, $l \rightarrow s$ for the solid-gas coexistence curve. We have made the assumption, as instructed, that $v_g >> v_l, v_s$ and that the vapor is an ideal gas.

We are given one point each on the coexistence curves on the $PT$-diagram: $(T_1, P_{10})$ and $(T_2, P_{20})$, where 1 and 2 refer to the liquid-gas and solid-gas curves, respectively. We are also given the latent heats $L_1$ and $L_2$. Both curves are governed by (21.1.4), the solution of which reads

\[
P_i(T) = P_{i0} \exp \left[ -\frac{L_i}{R} \left( \frac{1}{T} - \frac{1}{T_i} \right) \right].
\]

(21.1.5)

The triple point is where these two curves intersect:

\[
T = \frac{\frac{1}{R}(L_2 - L_1)}{\frac{1}{R}(\frac{T_2}{T_1} - \frac{T_1}{T_1}) - \ln \frac{P_{20}}{P_{10}}} \approx 196 \text{ K}, \quad P = 46 \text{ mm of Hg}.
\]

(21.1.6)
21.2 Cooling by Pumping

[Spring 2008 Modern (Morning), Problem 6] In my laboratory, we cool samples by immersing them in liquid $^4$He in a dewar. The dewar allows heat $Q$ per second to flow into the liquid, and to evaporate a corresponding volume of liquid. We assume that the latent heat of vaporization per mole, $L$, is independent of temperature.

At atmospheric pressure, $p_0$, liquid $^4$He boils at $T_0 = 4.2$ K; however, we often need to reduce the temperature, which we accomplish by lowering the pressure over the liquid by means of a pump. The pump is at room temperature, $T_R$. A volume $V$ of gas (as measured at the pump inlet) passes through the pump per second, regardless of the pressure of the gas, which we assume to be constant throughout the system. We also assume that the $^4$He vapor obeys the ideal gas law and that when it reaches the pump, it is at temperature $T_R$.

(a) Find an expression for the lowest pressure, $p_p$, that the pump can maintain over the surface of the liquid $^4$He in terms of $Q$, $T_R$, $L$, $V$ and the gas constant $R$.

(b) Use the Clausius-Clapeyron equation to find an expression for the corresponding temperature, $T_{He}$, of the liquid helium in thermal equilibrium with its vapor at pressure $p_p$. Express your answer in terms of $p_0$, $p_p$, $T_0$, $L$ and $R$.

(c) Estimate $T_{He}$ for $p_0 = 10^5$ Pa, $Q = 0.1$ W, $L = 100$ J/mol and $\dot{V} = 20$ liter/sec.

[Hint: The Clausius-Clapeyron equation is $\frac{dp}{dT} \approx \frac{L}{RT^2}$, where $L$ is the latent heat per molecule, and $v_g$ and $v_L$ are the volumes occupied by one molecule in the gaseous and liquid states, respectively. You may assume $v_L$ is negligible compared to $v_g$.]

SOLUTION:

(a) It is impossible to do this question exactly without knowing how the boiling temperature of $^4$He varies with pressure. But, if you look at a phase diagram for $^4$He, you will notice that the boiling temperature stays more or less constant over a wide range of pressures. Hence, we will assume it to be constant. Also, we will find in part (b) that the equilibrium temperature of the liquid $^4$He is lower than the boiling temperature $T_0$. Thus, to determine how many moles is evaporated by the absorption of $Q$, we would need to know the heat capacity as well as the latent heat. However, the specific heat near $T_0$ happens to be $\approx 30$ J/mol-K, which is smaller than $L$, and we will assume that we cannot stray too far away from $T_0$, so that the amount of heat needed to bring the helium to the boiling point is much smaller than the amount needed to actually boil the helium. Thus, we will say that $n = Q/L$ moles of helium are boiled away.

Since we have assumed constant pressure throughout, $p_p$ is the pressure of this gas even right before exiting the pump, where it also has volume $V$ and temperature $T_R$. Hence, the ideal gas law allows us to solve for $p_p$:

$$p_p = \frac{QRT_R}{LV} \quad (21.2.1)$$

(b) $L_1 = L/N_A$, where $N_A$ is Avogadro’s number, and we should recall that $R = N_A k$. We will neglect $v_L$. The ideal gas law says $v_g = kT/p$. Hence, Clausius-Clapeyron reads

$$\frac{dp}{dT} \approx \frac{pL}{RT^2}. \quad (21.2.2)$$
We separate variables, integrate $p$ from $p_0$ to $p_p$ and $T$ from $T_0$ to $T_{He}$, and solve for $T_{He}$:

$$T_{He} = \left(1 + \frac{RT_0}{L} \ln \frac{p_0}{p_p}\right)^{-1} T_0$$

(21.2.3)

(c) This is just plug and chug: $p_p = 122$ Pa and $T_{He} = 1.26$ K. You can see that the approximations we used are TERRIBLE! Something rather dramatic occurs between $T_0$ and $T_{He}$: the lambda point, below which $^4$He goes superfluid! The heat capacity goes through a discontinuity (lambda-looking; that’s where the name comes from) at around 2.17 K. The heat capacity at around 1.26 K is now on the order of 60 J/mol-K and it spikes right at the lambda point. Therefore, it seems rather silly to think that all of $Q$ is used up just for boiling. Unfortunately, taking into account the heat required to bring $^4$He to $T_0$ requires computers and numerical analysis.
21.3 Liquid-Vapor Carnot Cycle

[Lim 1114] A Carnot cycle is operated with liquid-gas interface. The vapor pressure is $p_v$, temperature $T$, volume $V$. The cycle goes from 1 to 2, evaporating $n$ moles of liquid. This is followed by a reversible cooling from 2 to 3 (with temperature change $-\Delta T$), then there is an isothermal contraction from 3 to 4 (at which point the pressure is $p_v - \Delta p$), recondensing $n$ moles of liquid, and finally a reversible heating from 4 to 1 (with temperature change $\Delta T$), completes the cycle.

(a) Let $v_g$ and $v_l$ be the molar volume of gas and liquid, respectively. Let $L_v$ be the latent heat of vaporization per mole. Assuming $\Delta p$ and $\Delta T$ are small, calculate the efficiency.

(b) Recognizing that any two Carnot engines operating between $T$ and $T - \Delta T$ must have the same efficiency (why?) and that this efficiency is a function of $T$ alone, use the result of part (a) to obtain an expression for $dp_v/dT$ thus “rederiving” the Clausius-Clapeyron relation.

**SOLUTION:**

(a) The difference between the volume at 2 and the volume at 1 is just the difference between the volume of $n$ moles of gas versus $n$ moles of liquid: $V_2 - V_1 = n(v_g - v_l)$. Since $\Delta p$ and $\Delta T$ are small, we can approximate the curvy “parallelogram” in the $p$–$V$ plane to a thin rectangle of length $V_2 - V_1$ and width $\Delta p$. Thus, the total work done by the engine, which is the area of this rectangle, is $W = n(v_g - v_l)\Delta p$. Meanwhile, the absorbed heat is $Q = nL_v$. Thus, the efficiency is

$$\eta = \frac{W}{Q} = \frac{(v_g - v_l)\Delta p}{L_v}. \quad (21.3.1)$$

(b) The Carnot efficiency is

$$\eta = \frac{\Delta T}{T}. \quad (21.3.2)$$

Setting this equal to (21.3.1) yields

$$\frac{dp_v}{dT} = \frac{\Delta p}{\Delta T} = \frac{L_v}{T(v_g - v_l)} \quad (21.3.3)$$

If you want, you can write $L_v = T\Delta s = T(s_g - s_l)$, which gives

$$\frac{dp_v}{dT} = \frac{s_g - s_l}{v_g - v_l} \quad (21.3.4)$$
21.4 Water-Steam Coexistence

The entropy of water at atmospheric pressure and 100°C is 0.31 cal/g·K, and the entropy of steam at the same temperature and pressure is 1.76 cal/g·K.

(a) What is the heat of vaporization at this temperature?

(b) The enthalpy of steam under these conditions is 640 cal/g. Calculate the enthalpy of water under these conditions.

(c) Calculate the Gibbs free energy of water and steam under these conditions.

SOLUTION:

(a) \[ L = T \Delta S = (273.15 \text{ K})(1.76 - 0.31 \text{ cal/g · K}) = 396 \text{ cal/g}. \]

(b) Since \( dH = TdS + Vdp \), \[ H_w = H_s - T \Delta S = 244 \text{ cal/g}. \]

(c) \[ G_w = H_w - T S_w = 159 \text{ cal/g} \] and \[ G_s = H_s - T S_s = 159 \text{ cal/g}. \]
21.5 Boiling Water

Based on “Physics for Entertainment” by Y. Perelman

(a) Put some pure water into an open glass bottle of negligible thickness (say, half-filled, for example). Immerse the bottle into a pot of boiling water such that the bottle is not in contact with the bottom of the pot. Will the water in the bottle boil? Explain.

(b) By some means, start the water in the small bottle boiling and almost immediately after cork it tightly. Even in the case when you are able to thermally isolate this system from the rest of the universe, explain why the boiling will eventually cease.

(c) In part (b), just after the boiling has ceased, you have a choice of pouring boiling water or cold water on the corked bottle in the hopes of restarting the boiling. Which would you choose and why?

SOLUTION:

(a) No. The water in the pot is at 100°C. Heat transfer into the water in the bottle will cease once it reaches 100°C, when it is at thermal equilibrium with the water in the pot. More heat needs to be transferred to the water in the bottle (latent heat of vaporization) in order to boil it.

(b) Below is a phase diagram for water.

![Phase diagram for water](image)

Figure 21.1: Phase diagram for water.

Since the bottle is corked, as more water boils off, the pressure increases. The phase diagram shows that the boiling temperature rises as pressure rises. Thus, eventually, the boiling will stop.

(c) Pouring boiling water will do absolutely nothing as evidenced by part (a). Pouring just enough cold water will condense the water vapor in the bottle. Since air was pushed out of the bottle when boiling occurred, there will now be an air deficit and the pressure will go down. If done just right, the decreased pressure will decrease the boiling temperature just enough to resume boiling briefly.
Chapter 22

Kinetic Theory

• Maxwell speed distribution: $P(v) = 4\pi \left( \frac{m}{2\pi kT} \right)^{3/2} v^2 e^{-mv^2/2kT}$.

• I find the discussion in Kittel of kinetics and propagation too cumbersome and prefer the brief section entitled “Rates of Processes” (section 1.7) in Schroeder despite the fact that Kittel’s discussion is more precise and general. My feeling is that Schroeder’s simpler and more basic treatment will suffice.

• Heat flow: $\dot{Q} = -KA \frac{dT}{dx}$, where $x$ is the direction along the heat flow. The sign is unnecessary as long as you know that heat flows from hot to cold objects.

• Heat equation in 1D: $Q(t) = cpT(x, t) \, dx$ heat in rod between $x$ and $x + dx$. Then, $Q(t + dt) - Q(t) = cp(T(x, t + dt) - T(x, t)) \, dx$, is equal to heat flowing in at $x$ minus heat flowing out at $x + dx$ in time $dt$, which is $dt(-K \partial_x T (\int_x^{x+dx} - |x|))$. Equating, and dividing by $dx \, dt$ gives $\partial_t T = \kappa \partial_x^2 T$ where $\kappa = K/cp$.

• Mean free path: $\ell \approx 1/4\pi r^2 n$ where $r$ is the typical particle radius and $n$ is the particle number density. Thermal conductivity: $K = \frac{1}{2} \bar{v} C_v \ell \bar{v}$, where $\bar{v}$ is the mean particle speed. Schroeder’s derivation is simplistic and in 1D. A more careful derivation in Kittel simply changes the $1/2$ to a $1/3$. Big whoop!

• Diffusion: $J_x = -D \partial_n / \partial x$ where $n$ is particle density. Fick’s law: $\dot{n} = D \nabla^2 n$. Again, in terms of mean free path, $D = \bar{v} \ell / 3$.

• Detailed balance: in thermal equilibrium, rate of $A \rightarrow B$ equals rate of $B \rightarrow A$. 
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CHAPTER 22. KINETIC THEORY

22.1 Insulated Pipe

[Luk, Fall 2005 MT1.3] A thin-walled cylindrical pipe is used to carry hot water at a temperature of 363 K. The diameter of the pipe is 2.54 cm and it is wrapped with a 2.54 cm thick layer of insulation whose thermal conductivity is 0.05 W/K·m. A section of the pipe, 5 m long, passes through a room, which is at temperature 283 K.

(a) What is the rate at which the hot water is losing heat through the insulation?

(b) What is the rate of change of entropy of the hot water? the insulation? the room? the universe? Treat the hot water and room as reservoirs whose temperature changes are negligible.

SOLUTION:

(a) Consider a differential bit of insulation $r\,dr\,d\phi\,dz$, where $r$ here is the cylindrical radius (not spherical). The cross-sectional area of this bit as seen from the central axis of the cylinder is $dA = r\,d\phi\,dz$. Let the temperature change from $r$ to $r + dr$ be $dT$. Then, the differential bit of heat flowing through this bit of insulation is

$$d\dot{Q} = -K\,dA\frac{dT}{dr} = -Kr\,d\phi\,dz\frac{dT}{dr}.$$  \hspace{1cm} (22.1.1)

Integrating over $\phi$ and $z$ yields

$$\dot{Q} = -2\pi LKr\frac{dT}{dr} \implies \dot{Q} = \frac{2\pi LK(T_{in} - T_{out})}{\ln(R_{out}/R_{in})} = 114 \text{ W}. \hspace{1cm} (22.1.2)$$

(b) $\dot{S}_{water} = -\dot{Q}/T_{water} = -0.314 \text{ W/K}$. In steady state, the heat that flows into the insulation must flow out into the room. Hence, $\dot{S}_{ins} = 0 \text{ W/K}$. For the room, $\dot{S}_{room} = \dot{Q}/T_{room} = 0.403 \text{ W/K}$. For the universe, we simply add all three: $\dot{S}_{uni} = 0.089 \text{ W/K}$. It’s a good thing that the latter is positive.
22.2 Heat Transport in a Wire

[Fall 2004 Modern (Morning), Problem 5] Consider a wire of length $L$ and whose radius varies linearly from one end to the other. One end has radius $r_A$ and temperature $T_A$ while the other end has radius $r_B$ and temperature $T_B$. The thermal conductivity is $K$.

(a) Find the rate of heat flow through the wire.

(b) Assume that $r_B = 2r_A$, that $T_A = 0^\circ C$ and $T_B = 90^\circ C$. Find the temperature at a point half-way down the wire, at $L/2$.

**SOLUTION:**

(a) Let $x$ be the position along the wire from $x = 0$, where $r = r_A$, to $x = L$, where $r = r_B$. Hence, $r(x) = r_A + \left(\frac{r_B - r_A}{L}\right)x$. The heat flow equation reads $\dot{Q} = -K\pi r^2 \frac{dT}{dx}$. At steady state $\dot{Q}$ is constant in space and time. Hence,

\[
\int_{T_A}^{T_B} dT = -\frac{\dot{Q}}{\pi K} \int_0^L \frac{dx}{r^2} = -\frac{\dot{Q}L}{\pi (r_B - r_A)K} \int_{r_A}^{r_B} \frac{dr}{r^2}
\]

\[
T_B - T_A = \frac{\dot{Q}L}{\pi (r_B - r_A)K} \frac{r_A - r_B}{r_A r_B} = -\frac{\dot{Q}L}{\pi r_A r_B K}. \quad (22.2.1)
\]

Therefore, solving for $\dot{Q}$ yields

\[
\dot{Q} = \frac{\pi r_A r_B K (T_A - T_B)}{L}. \quad (22.2.2)
\]

(b) Call the midway point, $C$. The same procedure yields

\[
T_C - T_A = -\frac{\dot{Q}L}{\pi r_A r_B K} \frac{r_B}{r_C} \left(\frac{r_C - r_A}{r_B - r_A}\right) = \alpha (T_B - T_A), \quad (22.2.3)
\]

where $\alpha \equiv \frac{r_B}{r_C} \left(\frac{r_C - r_A}{r_B - r_A}\right) = \frac{2}{3}$.

It follows that

\[
T_C = \frac{T_A + 2T_B}{3} = 333 \text{ K} = 60^\circ C. \quad (22.2.4)
\]
22.3 Speed of Sound in Air

[Lim 1020 & 1149] Neglecting viscosity and heat conductivity, small disturbances in a fluid propagate as undamped sound waves.

(a) Show that the sound wave speed is given by \( v^2 = \frac{dp}{d\rho} \), where \( p \) is pressure and \( \rho \) is mass density.

(b) Determine the speed of sound in an ideal gas for which the compressions and rarefactions are isothermal. Repeat for the adiabatic case.

(c) Which gives a better result for air?

SOLUTION:

(a) Describe the air by a density, pressure, and velocity field, \( \rho, p \text{ and } u \), all three of which are functions of space and time. Expand each one around some fixed background: \( \rho = \rho_0 + \rho' \), \( p = p_0 + p' \) and \( u = 0 + u' \), where we have set the background velocity field to zero since we want the speed of sound in still air (no wind, etc.)

The momentum density is \( j = \rho u \approx \rho_0 u' \). Fick’s law, or mass conservation, implies \( \dot{\rho} = -\nabla \cdot (\rho u) \). Plugging in the expansions and differentiating once more with respect to time yields

\[
\ddot{\rho} = -\rho_0 \nabla \cdot \dot{u}'. \tag{22.3.1}
\]

The force density due to pressure is \( f = -\nabla p \). Newton’s third law equates this with the time derivative of \( j \). Hence, \( \rho_0 \dot{u}' = -\nabla p' \). Taking the negative divergence of both sides yields

\[
-\rho_0 \nabla \cdot \dot{u}' = \nabla^2 p'. \tag{22.3.2}
\]

Combining (22.3.1) and (22.3.2) yields

\[
\ddot{\rho} = \nabla^2 p = \left( \frac{\partial p}{\partial \rho} \right) \nabla^2 \rho. \tag{22.3.3}
\]

The wave equation, \( \ddot{\rho} = v^2 \nabla^2 \rho \), implies \( v^2 = \frac{\partial p}{\partial \rho} \).

(b) The equation of state for the isothermal case is \( pV = \text{constant} \), whereas for the adiabatic case is \( pV^{\gamma} = \text{constant} \). Consider the general case, \( pV^x = \text{constant} \). We may write this as

\[
\frac{dp}{dV} = -x \frac{p}{V}. \tag{22.3.4}
\]

Let \( m \) and \( \nu \) be the molar mass and volume of the gas, so that \( \rho = m/\nu \). Then,

\[
-x \frac{p}{\nu} = \frac{dp}{d\nu} = \frac{dp}{d\nu} \frac{dp}{d\rho} = -\frac{m}{\nu^2} \frac{dp}{d\rho}. \tag{22.3.5}
\]

This gives the speed

\[
v^2 = \frac{dp}{d\rho} = \frac{x\nu}{m} = \frac{xRT}{m}. \tag{22.3.6}
\]

(c) \( v_T = \sqrt{RT/m} = 280 \text{ m/s} \) and \( v_\gamma = \sqrt{\gamma RT/M} = 331 \text{ m/s} \), where \( m = 28.8u \) was used for 80\% N\(_2\) and 20\% O\(_2\), and \( \gamma = 7/5 \) for diatomic molecules. We used \( T = 0^\circ \text{C} \). It appears that the adiabatic result is more reliable.
Part IV

Quantum Mechanics
Chapter 23

Time-Independent Schrödinger Equation

- \( \hat{H} \psi = E \psi \) where \( \hat{H} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V \).
- \( \Psi(x, t) = \psi(x) e^{-iEt/\hbar} \).
- Harmonic oscillator: \( V(x) = \frac{1}{2} m \omega^2 x^2 \).
  - Algebraic Method: \( a_\pm = (2\hbar m \omega)^{-1/2}(\mp ip + m \omega x) \) with \( [a_-, a_+] = 1 \). Number operator: \( \hat{N} = a_+ a_- \). Hamiltonian: \( \hat{H} = \hbar \omega (\hat{N} + \frac{1}{2}) \). Ground and excited states: \( a_- \psi_0 = 0 \) and \( \psi_n = \frac{1}{\sqrt{n!}} (a_+)^n \psi_0 \), with \( a_+ \psi_n = \sqrt{n+1} \psi_{n+1} \) and \( a_- \psi_n = \sqrt{n} \psi_{n-1} \).
  - \( x = \sqrt{\hbar/2m\omega}(a_+ + a_-) \) and \( p = i\sqrt{\hbar m \omega}/2(a_+ - a_-) \) to find expectation values of powers of \( x \) and/or \( p \).
  - Analytic Method: UGH!
- Fourier transform: \( f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(k) e^{ikx} dk \) and \( f(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \) for space and the opposite sign in the exponential for time Fourier transforms. [Note: usually, I will not put tildes or otherwise distinguish \( f \) and its Fourier transform by anything other than their arguments.]
- Wavefunction boundary conditions:
  - \( \psi \) is always continuous;
  - \( d\psi/dx \) is continuous except at points where the potential is infinite.
23.1 Time-Periodic Wavefunction

[Kevin G.] A system is governed by a real Hamiltonian and whose wavefunction is real at $t = 0$ and at a later time $t = t_1$. Show that the wavefunction is periodic in time. What is the period, $T$? Show that the energy must come in integer multiples of $2\pi\hbar/T$.

**SOLUTION:**

I will suppress the spatial coordinate. Then, $\psi(t_1) = e^{-iE_{t_1}/\hbar}\psi(0)$ and, complex conjugating, $\psi^*(t_1) = e^{iE_{t_1}/\hbar}\psi^*(0) = e^{iE_{t_1}/\hbar}\psi(0)$, where we used the fact that $\psi$ is real at $t = 0$ and $t = t_1$.

Instead of propagating forwards in time, let us propagate backwards: $\psi(0) = e^{iE_{t_1}/\hbar}\psi(t_1) = e^{2iE_{t_1}/\hbar}\psi(0)$, where the last step used the result of the previous paragraph. Complex conjugating this and using reality again gives $\psi(0) = e^{-2iE_{t_1}/\hbar}\psi(0)$. Therefore,

$$\psi(t) = e^{-iE_{t_1}/\hbar}\psi(0) = e^{iE_{t_1}/\hbar}e^{-2iE_{t_1}/\hbar}\psi(0) = e^{-iE(t+2t_1)/\hbar}\psi(0) = \psi(t + 2t_1).$$  \hspace{0.5cm} (23.1.1)

Hence, the wavefunction is periodic in time with period $[T = 2t_1]$.

Let the eigenstates of the time-independent Schrödinger equation be $\psi_j(x)$, with energies $E_j$. Then an arbitrary state may be expanded as $\psi(x,t) = \sum_j c_j e^{-iE_jt/\hbar}\psi_j(x)$. Then,

$$\psi(x,t+T) = \sum_j c_j e^{-iE_jT/\hbar}e^{-iE_jT/\hbar}\psi_n(x) = \psi(x,t)$$ \hspace{0.5cm} (23.1.2)

requires that $e^{-iE_jT/\hbar} = 1$ for all $j$. Hence,

$$E_j = 2\pi n_j \hbar/T$$ \hspace{0.5cm} (23.1.3)

for some integer $n_j$. 
23.2 Two-Step Potential

[Kevin G.] Let $V$ be a two-step potential that goes (from left to right) from $V_1 = 0$ to $V_2 > V_1$ to $V_3 > V_2$. Let $a$ be the width of the first step when $V = V_2$. Consider a system with energy $E > V_3$. Calculate the scattering matrix, which relates the coefficients of the outgoing waves (in both directions) in terms of the incoming waves (in both directions). For leftward scattering only, calculate the transmission coefficient.

**SOLUTION:**

Let region $j$ be when $V = V_j$ and define $k_j = \frac{1}{\hbar}\sqrt{2m(E-V_j)}$, which is real and positive since $E > V_j$ for all $j$. Schrödinger’s equation in region $j$ reads $(\frac{d^2}{dx^2} + k_j^2)\psi_j(x) = 0$ whose solution is $A_je^{ik_jx} + B_je^{-ik_jx}$, where $A$ is right-moving and $B$ is left-moving.

Let region 2 extend from $x = -a$ to $x = 0$. Since the potential is finite everywhere, the wavefunction and its derivative must be continuous everywhere. This implies

$$A_1e^{-ik_1a} + B_1e^{ik_1a} = A_2e^{-ik_2a} + B_2e^{ik_2a}, \quad (23.2.1a)$$

$$k_1A_1e^{-ik_1a} - k_1B_1e^{ik_1a} = k_2A_2e^{-ik_2a} - k_2B_2e^{ik_2a}, \quad (23.2.1b)$$

$$A_2 + B_2 = A_3 + B_3, \quad (23.2.1c)$$

$$k_2A_2 - k_2B_2 = k_3A_3 - k_3B_3. \quad (23.2.1d)$$

We can use (23.2.1a) and (23.2.1b) to solve for $A_2$ and $B_2$ in terms of $A_1$ and $B_1$ or $A_3$ and $B_3$. This is summarized in matrix form as

$$\begin{pmatrix} A_2 \\ B_2 \end{pmatrix} = \frac{1}{2k_2} \begin{pmatrix} (k_1 + k_2)e^{-i(k_1-k_2)a} & -(k_1 - k_2)e^{i(k_1+k_2)a} \\ -(k_1 - k_2)e^{-i(k_1+k_2)a} & (k_1 + k_2)e^{i(k_1-k_2)a} \end{pmatrix} \begin{pmatrix} A_1 \\ B_1 \end{pmatrix}, \quad (23.2.2a)$$

$$\begin{pmatrix} A_2 \\ B_2 \end{pmatrix} = \frac{1}{2k_2} \begin{pmatrix} k_2 + k_3 & k_2 - k_3 \\ k_2 - k_3 & k_2 + k_3 \end{pmatrix} \begin{pmatrix} A_3 \\ B_3 \end{pmatrix}. \quad (23.2.2b)$$

Setting these two equal to each allows us to solve for $A_3$ and $B_3$ in terms of $A_1$ and $B_1$. After a lot of tedious algebra, we find $(^{A_3}_{B_3}) = M(^{A_1}_{B_1})$, where

$$M_{11} = M_{22}^* = \frac{1}{2k_2}e^{-ik_1a}[(k_1 + k_3)k_2 \cos(k_2a) + i(k_1k_3 + k_2^2) \sin(k_2a)], \quad (23.2.3a)$$

$$M_{12} = M_{21}^* = -\frac{1}{2k_2}e^{ik_1a}[(k_1 - k_3)k_2 \cos(k_2a) + i(k_1k_3 - k_2^2) \sin(k_2a)]. \quad (23.2.3b)$$

$M$ is called the transfer matrix and relates the coefficients on the right of the region where $V$ is changing to the coefficients on the left. We want $(^{B_3}_{A_3}) = S(^{A_1}_{B_1})$. The two are related by simply writing out the equation involving $M$ and solving for $B_1$ and $A_3$ in terms of $A_1$ and $B_3$ (in that order) instead. One finds

$$S = \frac{1}{M_{22}} \begin{pmatrix} -M_{21} & 1 \\ \det M & M_{12} \end{pmatrix}, \quad (23.2.3)$$

After some algebra, one finds det $M$ to be

$$\det M = k_1/k_3. \quad (23.2.4)$$

For leftward scattering, $B_3 = 0$, and the transmission coefficient is

$$T_l = \frac{|A_3|^2}{|A_1|^2} = |S_{21}|^2 = \frac{4k_1^2k_2^2}{(k_1 + k_3)^2k_2^2 \cos^2(k_2a) + (k_1k_3 + k_2^2)^2 \sin^2(k_2a)}. \quad (23.2.5)$$
23.3 Ramsauer Effect

[Basdevant & Dalibard 3.6] In 1921, Ramsauer noticed that for some particular values of the incident energy, rare gases such as helium, argon or neon were transparent to low-energy electron beams. This can be explained in the following one-dimensional model. Consider a stationary incident energy, rare gases such as helium, argon or neon were transparent to low-energy electron beams. This can be explained in the following one-dimensional model. Consider a stationary

\[ V(x) = \begin{cases} 0, & |x| > a, \\ -V_0, & |x| \leq a. \end{cases} \]  

(23.3.1)

(a) Calculate the transmission and reflection probabilities, \( T \) and \( R \), for leftward scattering. Verify that \( R + T = 1 \).

(b) Show that \( T = 1 \) for some values of the energy. Interpret this result and the Ramsauer effect.

(c) In helium, the lowest energy at which the phenomenon occurs is \( E = 0.7 \text{ eV} \). Assuming that the radius of the atom is \( a = 0.1 \text{ nm} \), calculate the depth \( V_0 \) of the potential well inside the atom in this model.

**SOLUTION:**

(a) Set \( k^2 = 2mE/\hbar^2 \) and \( q^2 = 2m(E + V_0)/\hbar^2 \), and denote the regions by 1 \( (x < -a) \), 2 \( (|x| \leq a) \) and 3 \( (x > a) \). For leftward scattering,

\[ \psi_1(x) = e^{ikx} + re^{-ikx}, \quad \psi_2(x) = Ae^{iqx} + Be^{-iqx}, \quad \psi_3(x) = te^{ikx}. \]  

(23.3.2)

Continuity of \( \psi \) and \( \psi' \) at the boundaries implies

\[ \begin{pmatrix} e^{-ika} & e^{ika} \\ ke^{-ika} & -ke^{ika} \end{pmatrix} \begin{pmatrix} 1 \\ r \end{pmatrix} = \begin{pmatrix} e^{-iqa} & e^{iqa} \\ qe^{-iqa} & -qe^{iqa} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}, \]  

(23.3.3a)

\[ \begin{pmatrix} 1 \\ t e^{ika} \end{pmatrix} = \begin{pmatrix} e^{iqa} & e^{-iqa} \\ qe^{iqa} & -qe^{-iqa} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}. \]  

(23.3.3b)

Solving for \( A \) and \( B \) using the first and the second equation, and then setting these equal to each other yields

\[ \begin{pmatrix} (k + q)e^{i(k-q)a} \\ -(k - q)e^{i(k+q)a} \end{pmatrix} t = \begin{pmatrix} (k + q)e^{-i(k-q)a} \\ -(k - q)e^{-i(k+q)a} \end{pmatrix} \begin{pmatrix} 1 \\ r \end{pmatrix}. \]  

(23.3.4)

If we multiply by the inverse of the large right-hand matrix, then the first entry of the resulting equation gives 1 in terms of \( t \), which can be inverted to solve for \( t \). Also, the two equations may be divided to get rid of \( t \) and solve for \( r \). Defining the quantity \( \Delta = (k^2 + q^2) \sin(2qa) + 2ikq \cos(2qa) \), the result may be written

\[ r = \frac{(k^2 - q^2) \sin(2qa)}{\Delta e^{2ika}}, \quad t = \frac{2ikq}{\Delta e^{2ika}}. \]  

(23.3.5)

Now, \(|\Delta|^2 = 4k^2q^2 + (k^2 - q^2)^2 \sin^2(2qa) \) and so

\[ R = |r|^2 = \frac{(k^2 - q^2)^2}{|\Delta|^2} \sin^2(2qa), \quad T = |t|^2 = \frac{4k^2q^2}{|\Delta|^2}. \]  

(23.3.6)

Indeed, we have \( R + T = 1 \).
(b) \( R = 0 \) (and so \( T = 1 \)) when \( 2qa = n\pi \) for \( n \in \mathbb{Z} \) or

\[
E_n = \frac{n^2\pi^2\hbar^2}{8ma^2} - V_0 ,
\]

(23.3.7)

The de Broglie wavelength of the particle inside the well is \( \lambda = \frac{2\pi}{q} \) and so perfect transmission occurs when the well size, \( 2a \), is an integer multiple of \( \lambda/2 \). We can interpret this classically like a thin-film interference problem where there is either a \( \lambda/2 \) phase shift at the first interface, or at the second, but not both (due to the Stoke’s relations, c.f. Chapter 10). Indeed, the minimal thickness of a film that produces destructively interfering reflected waves is \( \lambda/2 \) in this case.

(c) Set \( E_1 = 0.7 \text{ eV} \), \( n = 1 \), and \( a = 0.1 \text{ nm} \). This gives \( V_0 = 8.7 \text{ eV} \).
Chapter 24

Formalism

- Hilbert space: \( \mathcal{H} = \mathcal{L}^2(a, b) \), the space of square-integrable functions in the interval \((a, b)\), is the one mainly used in quantum mechanics.

- Inner product: \( \langle f | g \rangle = \int_a^b f(x)^* g(x) \, dx \) is the usual definition. However, an arbitrary positive function may be inserted in the integral and it would remain an inner product (positive definite sesquilinear form).

- Orthonormality: \( \langle f_m | f_n \rangle = \delta_{mn} \).

- Completeness: \( \sum_m |f_m\rangle \langle f_m| = \mathbb{1} \). The sum could include an integral.

- Observables: Hermitian operators, \( \hat{Q} = \hat{Q}^\dagger \).

- Eigenvalues of Hermitian operators are real and eigenfunctions having distinct eigenvalues are orthogonal.

- Even eigenfunctions belonging to a degenerate eigenstate may be orthonormalized using the Gram-Schmidt procedure.

- When \( |\psi\rangle = \sum_m |f_m\rangle \langle f_m|\psi\rangle = \sum_m c_m |f_m\rangle \), then \( |c_m|^2 \) is the probability of observing the system in the eigenstate \( |f_m\rangle \). The sum may include integrals.

- Uncertainty principle: \( \sigma_A^2 \sigma_B^2 \geq \left( \frac{1}{2} \langle [\hat{A}, \hat{B}] \rangle \right)^2 \).

- Time evolution: \( \frac{d}{dt} \langle Q \rangle = \frac{i}{\hbar} \langle [\hat{H}, \hat{Q}] \rangle + \langle \frac{\partial Q}{\partial t} \rangle \).
24.1 Sequential Measurements

[Fall 2005 Modern (Afternoon), Problem 2; Griffiths 3.27] An operator \( \hat{A} \), representing observable \( A \), has two normalized eigenstates \( \psi_1 \) and \( \psi_2 \), with eigenvalues \( a_1 \) and \( a_2 \), respectively. Operator \( \hat{B} \), representing observable \( B \), has two normalized eigenstates \( \phi_1 \) and \( \phi_2 \), with eigenvalues \( b_1 \) and \( b_2 \). The eigenstates are related by

\[ \psi_1 = \frac{3\phi_1 + 4\phi_2}{5}, \quad \psi_2 = \frac{4\phi_1 - 3\phi_2}{5}. \quad (24.1.1) \]

(a) Observable \( A \) is measured, and the value \( a_1 \) is obtained. What is the state of the system (immediately) after this measurement?

(b) If \( B \) is now measured, what are the possible results, and what are their probabilities?

(c) Right after the measurement of \( B \), \( A \) is measured again. What is the probability of getting \( a_1 \)? (Note that the answer would be quite different if I had told you the outcome of the \( B \) measurement.)

SOLUTION:

(a) The state must be \( \psi_1 \).

(b) \( b_1 \) (prob. 9/25) or \( b_2 \) (prob. 16/25)

(c) We must solve for \( \phi_1 \) and \( \phi_2 \) in terms of \( \psi_1 \) and \( \psi_2 \):

\[ \phi_1 = \frac{3\psi_1 + 4\psi_2}{5}, \quad \phi_2 = \frac{4\psi_1 - 3\psi_2}{5}. \quad (24.1.2) \]

 Hence, the probability of getting \( a_1 \) is

\[ P(a_1) = (9/25)^2 + (16/25)^2 = 337/625 = 0.5392. \quad (24.1.3) \]
24.2 Neutrino Oscillations

(Spring 2009 Modern (Afternoon), Problem 5) We now know that neutrinos have mass so we can talk about their rest frame. In a simple model with only two neutrino flavors (electrons and muon neutrino), neutrino flavor states do not have definite mass and the electron neutrino has equal amplitude for having mass \( m_1 \) and \( m_2 \) and is the symmetric state. Suppose we have an electron neutrino at \( t = 0 \).

(a) What is the probability that the neutrino will be an electron neutrino at arbitrary time \( t > 0 \). 

(b) How long does it take for the electron neutrino to change completely into a muon neutrino?

(c) What is the probability of measuring a mass of \( m_1 \) as a function of time?

**SOLUTION:**

(a) Let \( |e⟩ \) and \( |m⟩ \) be the flavor basis (electron and muon, respectively), and let \( |1⟩ \) and \( |2⟩ \) be the mass basis. We are told that \( |e⟩ = \frac{1}{\sqrt{2}} (|1⟩ + |2⟩) \). The initial condition is \( |ψ(0)⟩ = |e⟩ \). Thus, in the neutrino rest frame, where the energy is entirely rest mass,

\[
|ψ(t)⟩ = \frac{1}{\sqrt{2}} (e^{-im_1c^2t/\hbar} |1⟩ + e^{-im_2c^2t/\hbar} |2⟩). \tag{24.2.1}
\]

Thus, \( |ψ(t)⟩ = \frac{1}{2} [e^{-im_1c^2t/\hbar} + e^{-im_2c^2t/\hbar}] \) and thus, the probability that the neutrino is of electron flavor is

\[
P_e(t) = |⟨e|ψ(t)⟩|^2 = \cos^2 \left( \frac{(m_2 - m_1)c^2t}{2\hbar} \right). \tag{24.2.2}
\]

(b) \( P_m(t) = 1 - P_e(t) = \sin^2 \left( \frac{(m_2 - m_1)c^2t}{2\hbar} \right) \). For the particle to be entirely of muon flavor, this must reach 1, or, in other words, \( \frac{(m_2 - m_1)c^2t}{2\hbar} = \frac{π}{2} \). Solving for \( t \) gives

\[
t = \frac{\pi \hbar}{(m_2 - m_1)c^2}. \tag{24.2.3}
\]

(c) \( |ψ(t)⟩ = \frac{1}{\sqrt{2}} e^{-im_1c^2t/\hbar} \), and so the probability of measuring a mass \( m_1 \) is \( P_1(t) = 1/2 \).
24.3 Harmonic Oscillator Coherent States

[Kevin G. and Griffiths 3.35] Coherent states of the harmonic oscillator saturate the uncertainty bound (just as the ground state does, but none of the excited states do.)

(a) Prove that the raising operator has no normalizable eigenfunctions, whereas the lowering operator does. A coherent state, $|\alpha\rangle$, is one such that $a_- |\alpha\rangle = \alpha |\alpha\rangle$, where $\alpha \in \mathbb{C}$.

(b) Calculate $\langle x \rangle$, $\langle x^2 \rangle$, $\langle p \rangle$ and $\langle p^2 \rangle$ in the state $|\alpha\rangle$.

(c) Find $\sigma_x$ and $\sigma_p$, and show that the uncertainty is saturated.

(d) Show that the coefficients in the expansion of $|\alpha\rangle$ in terms of energy eigenstates are $c_n = \alpha^n c_0 / \sqrt{n!}$ and determine $c_0$ by normalizing $|\alpha\rangle$.

(e) Now put in the time dependence and show that $|\alpha(t)\rangle$ remains an eigenstate of $a_-$, but with a time-evolving eigenvalue $\alpha(t) = e^{-i\omega t} \alpha$

(f) Is the ground state itself a coherent state? If so, what is the eigenvalue?

**SOLUTION:**

(a) By completeness, a normalizable state, $|\alpha\rangle$, can be expanded in terms of energy eigenstates: $|\alpha\rangle = \sum_{n=0}^{\infty} c_n |n\rangle$. There will necessarily be a lowest $n$ for which $c_n \neq 0$. Then, $|n\rangle \not\subset a_+ |\alpha\rangle$ and thus $|\alpha\rangle$ cannot be an eigenfunction of $a_+$.

$a_-$ has eigenfunctions by construction below.

(b) Let us assume that $|\alpha\rangle$ has been normalized. Since $a_+$ is the hermitian conjugate of $a_-$, we have $\langle \alpha | a_+ = (a_- | \alpha\rangle)^\dagger = (\alpha | \alpha\rangle)^\dagger = \alpha^* \langle \alpha |$. Using

$$x = \sqrt{\hbar/2m\omega} (a_+ + a_-), \quad p = i\sqrt{\hbar m\omega/2} (a_+ - a_-),$$

(24.3.1)

and $[a_-, a_+] = 1$, we find that

$$\langle x^2 \rangle = \langle \alpha | a_+ + a_- | \alpha\rangle = \langle \alpha^2 \rangle = (\hbar/2m\omega) [1 + (\alpha + \alpha^*)^2],$$

(24.2.2a)

$$\langle p^2 \rangle = -i\sqrt{\hbar m\omega/2} (\alpha - \alpha^*) = (\hbar m\omega/2) [1 - (\alpha - \alpha^*)^2].$$

(24.2.2b)

(c) $\sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2 = h/2m\omega$ and $\sigma_p^2 = \langle p^2 \rangle - \langle p \rangle^2 = h m\omega/2$. Thus, their product, $\sigma_x \sigma_p = h/2$, saturates the uncertainty bound.

(d) $a_- |\alpha\rangle = \sum_{n=1}^{\infty} c_n \sqrt{n} |n - 1\rangle = \sum_{n=0}^{\infty} c_{n+1} \sqrt{n+1} |n\rangle = \alpha |\alpha\rangle = \sum_{n=0}^{\infty} \alpha c_n |n\rangle$. This gives the recurrence relation $c_{n+1} = \alpha c_n / \sqrt{n+1}$, whose solution is $c_n = \alpha^n c_0 / \sqrt{n!}$, as is easily proven by induction.

We require $\sum_{n=0}^{\infty} |c_n|^2 = |c_0|^2 \sum_{n=0}^{\infty} \frac{|c_n|^2}{n!} = |c_0|^2 e^{\alpha^2} = 1$. Choosing $c_0$ to be real, we have $|c_0| = e^{-|\alpha|^2/2}$.

(e) $|\alpha(t)\rangle = \sum_{n=0}^{\infty} c_n e^{-|\alpha|^2/2} e^{-i(n+\frac{1}{2})\omega t} |n\rangle = e^{-i\omega t/2} \sum_{n=0}^{\infty} \left( \frac{\alpha e^{-i\omega t}}{\sqrt{n!}} \right) e^{-|\alpha|^2/2} |n\rangle$. The overall phase does not affect eigenfunction-ness.

(f) Yes, with eigenvalue 0.
Chapter 25

Quantum Mechanics in Three Dimensions

- Separation of variables for spherical potential: \( \psi(r, \theta, \phi) = R(r)Y(\theta, \phi) \) as solution to the spherical Laplacian: \( \nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} - \frac{\sin \theta}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \), where \( \hat{L}^2 = -\hbar^2 \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right) \), is the square of the orbital angular momentum operator.

- Angular equation: \( \hat{L}^2 Y(\theta, \phi) = \ell(\ell + 1)\hbar^2 Y(\theta, \phi) \). The solutions are spherical harmonics, \( Y^\ell_m(\theta, \phi) \).

- Important orthonormality condition: \( \int_0^\pi \sin \theta \, d\theta \int_0^{2\pi} d\phi (Y^\ell_m \ast Y^{\ell'}_{m'}) = \delta_{\ell\ell'} \delta_{mm'} \).

- Radial equation: \( V_{\text{eff}} = V + \frac{\hbar^2 \ell(\ell+1)}{2m} \) and \( u(r) \equiv r R(r) \), then \( \frac{d^2}{dr^2} + \frac{(\ell^2 + k(r)^2)}{r} u = 0 \), where \( k(r)^2 = 2m(E - V_{\text{eff}}(r))/\hbar^2 \).

- Hydrogen: \( \psi_{n\ell m} = R_n(r) Y^\ell_m(\theta, \phi) \), where \( n = 1, 2, \ldots \), \( \ell = 0, 1, \ldots, n-1 \) and \( m = -\ell, \ldots, \ell \). For \( \ell \neq 0 \), \( \psi \to 0 \). The number of radial nodes is \( n-1 \).

- \( R_n \sim e^{-x/n} x^n f_{n-\ell-1}(x) \), where \( x = r/a \) and \( a \) is the appropriate Bohr radius, and \( f_{n-\ell-1}(x) \) is some degree \( n - \ell - 1 \) polynomial in \( x \).

- Bohr energy: \( E_n = E_1/n^2 \) where \( E_1 = -\frac{1}{2} m (\alpha c)^2 = -13.6 \, \text{eV} \), where \( \alpha = e^2/4\pi \epsilon_0 \hbar c \) is the fine-structure constant, with \( \alpha \approx 1/137 \).

- Angular momentum: \( [L_x, L_y] = i\hbar L_z \) (and cyclic permutations). \( L_\pm \equiv L_x \pm i L_y \), then \( L_\pm Y^\ell_m = \hbar \sqrt{\ell(\ell+1)} - m(m \pm 1) Y^{\ell \pm 1}_m \).

- Spin-1/2: \( \hat{S} = \frac{\hbar}{2} \sigma \) where
  \[
  \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
  \]

- Magnetic dipole moment: \( \hat{\mu} = \gamma \hat{S} \), where \( \gamma \) is the gyromagnetic ratio. You should remember that \( \gamma \) is negative for an electron.

- Hamiltonian for spin in magnetic field: \( \hat{H} = -\hat{\mu} \cdot \mathbf{B} \).

- Addition of angular momentum: label states by total angular momentum and its z-component. A tensor product state is expanded in terms of states with allowed total angular momentum, \( \ell = |\ell_1 - \ell_2|, \ldots, \ell_1 + \ell_2 \), with coefficients that may depend on \( \ell, \ell_1, \ell_2, m_1 \) and \( m_2 \) (Clebsch-Gordan coefficients).
25.1 Angular Momentum Commutation Relations

[Basdevant and Dalibard 10.5]

(a) Show that \([J_x^2, J_y^2] = [J_y^2, J_z^2] = [J_z^2, J_x^2]\).

(b) Show that these three commutators vanish in states where \(j = 0\) or \(j = 1/2\).

(c) Show that these commutators also vanish in states where \(j = 1\). Find the common eigenbasis of \(J_x^2, J_y^2\) and \(J_z^2\) in this case.

SOLUTION:

(a) \([J^2, J_k] = \sum_j \{[J_j, J_k], J_j\} = i\hbar \sum_j \epsilon_{ijk} \{J_i, J_j\} = 0\) since \(\epsilon_{ijk}\) is antisymmetric in \(ij\), whereas \(\{J_i, J_j\}\) is symmetric. Obviously, then, \([J^2, J_k^2] = 0\) as well. Since \(J_k\) obviously commutes with itself, \([J^2 - J_k^2, J_k^2] = 0\). Choose, for example, \(k = z\), then \([J_x^2, J_y^2] + [J_y^2, J_z^2] = 0\), which yields the last equality. The first is found by choosing \(k = y\).

(b) The state with \(j = 0\) has \(J_x, J_y, J_z = 0\) as well and so the result is trivial. For \(j = 1/2\), we have \(J = (\hbar/2)\sigma\) and so \(J_i^2 = (\hbar/2)^2 \sigma_i^2 = (\hbar/2)^2 \mathbf{1}\), since the Pauli matrices square to the identity. Obviously the \(J_i^2\) all mutually commute.

(c) Take, for example, \(\langle 1, m_2 | [J_x^2, J_z^2] | 1, m_1 \rangle = (m_1^2 - m_2^2) \langle 1, m_2 | J_x^2 | 1, m_1 \rangle\). The case \(m_1 = m_2\) vanishes trivially. By symmetry, we need only consider the case \(m_1 = 0\) and \(m_2 = \pm 1\). From \(J_x = \frac{1}{2}(J_+ + J_-)\), we have \(J_x | 1, 0 \rangle \propto (|1, 1 \rangle + |1, -1 \rangle)\) and \(J_x | 1, \pm 1 \rangle \propto |1, 0 \rangle\), which implies \(J_z \propto |1, 0 \rangle\), and thus has vanishing inner product with \(|1, \pm 1 \rangle\).

As argued above, \(|1, 0 \rangle\) is an eigenstate of all three \(J_i^2\) operators. We must make two more out of the remaining states, \(|1, \pm 1 \rangle\). By symmetry, the other two are the states \(|1, 0 \rangle\) in the \(x\) and \(y\) bases. So, the eigenbasis is

\[
|1, 0 \rangle\] and \[
\frac{1}{\sqrt{2}}(|1, 1 \rangle \pm |1, -1 \rangle)
\] (25.1.1)
25.2. Rigid Rotator Angular Momentum

[Fall 1992 Modern (Morning), Problem 1] You are given that the state at \( t = 0 \) of a free rigid rotator, whose moment of inertia is \( I \), is specified by \( \psi(\theta, \phi, t = 0) = C(Y^1_0 - 2Y^1_8 - 2Y^1_9) \), where \( Y^m_\ell(\theta, \phi) \) represents its \((\ell, m)\)th spherical harmonic wavefunction, and \((\theta, \phi)\) represents the orientation of the rotator axis.

(a) Calculate the normalization constant, \( C \).

(b) Calculate \( \langle L^2 \rangle \) and \( \langle L_z \rangle \) in this state.

(c) Calculate \( \langle L_z^2 \rangle \) in this state.

(d) Find the state at time \( t \neq 0 \) and recalculate part (b).

**SOLUTION:**

(a) The spherical harmonics are orthonormal. Therefore, \( C^2(1 + 2^2 + 2^2) = 1 \) implies \( C = 1/3 \).

(b) Define \( \langle \cdot \rangle^m_\ell \equiv \langle \ell, m | \cdot | \ell, m \rangle \) so that \( \langle L^2 \rangle^m_\ell = \hbar^2 \ell(\ell + 1) \) and \( \langle L_z \rangle^m_\ell = m \hbar \). Since \( L^2 \) and \( L_z \) are diagonal in this basis, there are no cross terms for either operator between unlike spherical harmonics. Thus,

\[
\langle L^2 \rangle = \frac{\hbar^2}{9} [9(10) + 2^2(8)(9) + 2^2(9)(10)] = 82\hbar^2,
\]

\[
\langle L_z \rangle = \frac{4}{9} [1 + 2^2(-1) + 2^2(3)] = \hbar.
\]  \( \Box \)

(c) We write \( L_x = \frac{1}{2}(L_+ + L_-) \) and so \( L_x^2 = \frac{1}{4}(L_+^2 + L_+ L_- + L_- L_+ + L_-^2) \). The middle two terms link each spherical harmonic to itself and the outer two terms only link the two states with \( \ell = 9 \). We must recall \( L_\pm Y^m_\ell = \hbar \sqrt{\ell(\ell + 1) - m(m \pm 1)} Y^m_{\ell \pm 1} \). From this point on, this is just a lot of algebra, so just make sure you understand what you’re doing.

Let’s be a bit more clever, however. We can also write \( L_y = \frac{1}{2\imath}(L_+ - L_-) \) and so \( L_y^2 = \frac{1}{4}(L_+^2 + L_+ L_- + L_- L_+ - L_-^2) \). We see that \( L_x^2 + L_y^2 = \frac{1}{2}(L_+ L_- + L_- L_+) \). But, \( L_+^2 + L_-^2 = L^2 - L_z^2 \) and thus we may write \( L_x^2 = \frac{1}{4}(L_+^2 + L_-^2) + \frac{1}{2}(L^2 - L_z^2) \). We already calculated \( \langle L^2 \rangle \) and \( \langle L_z \rangle \) in part (b)! We have \( \langle L_x^2 \rangle = 4I\hbar^2/9 \).

\( \langle L_x^2 \rangle = \frac{2}{9}(9,3)|L_x^2 \cdot \frac{1}{3}|9,1 \rangle = \frac{8}{9} \sqrt{2 \cdot 3 \cdot 7 \cdot 11} \hbar^2 \approx 19.1 \hbar^2 \) and it turns out that \( \langle L_x^2 \rangle \) is the same. Hence,

\[
\langle L_x^2 \rangle = \left[ \frac{1}{4}(2 \cdot 19.1) + \frac{1}{2}(82 - \frac{41}{9}) \right] \hbar^2 = 48.3 \hbar^2.
\]  \( \Box \)

(d) We have \( H = L^2/2I \), so \( \psi(t) = \frac{1}{3} \left[ Y^1_9 e^{-45\hbar t/I} - 2Y^1_8 e^{-36\hbar t/I} + 2Y^1_9 e^{-45\hbar t/I} \right] \). \( L_x^2 \) and \( L_z \) commute with \( H \) and are thus conserved. So, (25.2.1) is unchanged.
25.3 Oxygen-Copper-Oxygen Spin Interactions

[Spring 1993 Modern (Afternoon), Problem 2] In a simplified model for defects in high temperature superconductors, consider two spin-1/2 oxygen ions on either side of a spin-1/2 copper ion. Assume that the three spins interact with one another according to the antiferromagnetic Heisenberg Hamiltonian

\[ H = J \mathbf{S}_1 \cdot \mathbf{S}_3 + K (\mathbf{S}_1 + \mathbf{S}_3) \cdot \mathbf{S}_2, \]

where \( \mathbf{S}_1 \) and \( \mathbf{S}_3 \) are the oxygen spins and \( \mathbf{S}_2 \) is the copper spin. \( J \) is positive, favoring antiparallel alignment of the oxygen spins.

(a) Does \( \mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3 \) commute with the Hamiltonian?

(b) For \( K = 0 \), solve for the complete spectrum. Be sure to state the degeneracies of each energy.

(c) What is(are) the ground state wavefunction(s)? Express your answer(s) as linear combinations of the states \( |m_1, m_2, m_3\rangle \), where \( m_i = \pm 1/2 \) is the \( z \)-component of the spin of spin \( i \).

(d) For \( K = J \), determine the complete spectrum. Again, be sure to state the degeneracies of each energy.

(e) Again, what is(are) the ground state wavefunction(s)?

SOLUTION:

(a) You could compute this by brute force using \[ [\mathbf{S}_i, \mathbf{S}_j \times \mathbf{S}_k] = i \hbar \delta_{ij} \epsilon_{\mu \nu \lambda} S_\lambda, \] where \( S_\mu \) is the \( \mu \)-th component of the \( i \)-th spin. Or, you could simply note that \( \mathbf{S} \) generates simultaneous rotations of all three spins and thus leaves their mutual dot products invariant, thus implying that it commutes with \( H \).

(b) We may write \[ H = \frac{J}{2} \left[ (\mathbf{S}_1 + \mathbf{S}_3)^2 - S_1^2 - S_3^2 \right] + \frac{K}{2} \left[ (\mathbf{S}_1 + \mathbf{S}_3)^2 - \frac{3h^2}{2} \right]. \] The possible values of the total spin of the two spin-1/2 oxygen ions are 0 and 1. The singlet energy is \( E_{\text{singlet}} = -3h^2J/4 \) and is 2-degenerate, while the triplet energy is \( E_{\text{triplet}} = h^2J/4 \) and is 6-degenerate (note: the copper spin degeneracy is 2.)

(c) There are two ground state wavefunctions:

\[ \frac{1}{\sqrt{2}} (|\uparrow \uparrow \text{ or } \downarrow \downarrow\rangle - |\downarrow \uparrow \text{ or } \uparrow \downarrow\rangle). \] (25.3.1)

(d) We may write \[ H = \frac{J}{2} \left[ S_\mu^2 - S_\mu^2 - S_\mu^2 - S_\mu^2 \right] + \frac{K}{2} \left[ S_\mu^2 - \frac{3h^2}{4} \right]. \] The possible values of \( S \) are 1/2 (two of these) and 3/2. The doublet energy is \( E_{\text{doublet}} = -3h^2J/4 \) and is 4-degenerate (since there are two doublets), and the quadruplet energy is \( E_{\text{quadruplet}} = 3h^2J/4 \) and is 4-degenerate.

(e) \[ |\frac{1}{2}, \frac{1}{2}\rangle_2 \] can be the answer to part (c). Then, \[ |\frac{1}{2}, \frac{1}{2}\rangle_2 = \frac{1}{\sqrt{6}} (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle - 2|\uparrow\uparrow\rangle) \] and its descendent.
25.4 Electron in Oscillating Magnetic Field

[Griffiths 4.33] An electron is at rest in an oscillating magnetic field \( \mathbf{B} = B_0 \cos(\omega t) \hat{\mathbf{k}} \), where \( B_0 \) and \( \omega \) are constants.

(a) Construct the Hamiltonian matrix for this system.

(b) The electron starts out (at \( t = 0 \)) in the spin-up state with respect to the \( x \)-axis. Determine the spin state \( \chi(t) \) at any subsequent time. \textit{Beware:} This is a time-dependent Hamiltonian, so you cannot get \( \chi(t) \) in the usual way from stationary states. Fortunately, in this case, you can solve the time-dependent Schrödinger equation directly.

(c) Find the probability of getting \( -\hbar/2 \), if you measure \( S_x \).

(d) What is the minimum field \( (B_0) \) required to force a complete flip in \( S_x \)?

\textbf{SOLUTION:}

(a) \( \hat{H} = -\gamma \mathbf{B} \cdot \hat{\mathbf{S}} = -\frac{\gamma B_0 \hbar}{2} \cos(\omega t) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -\frac{\gamma B_0 \hbar}{2} \cos(\omega t) \left[ |\uparrow\rangle \langle \uparrow| - |\downarrow\rangle \langle \downarrow| \right] \).

(b) The state that is spin up in \( x \) is \( \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle) \). Thus, let our state by \( \chi(t) = \alpha(t) |\uparrow\rangle + \beta(t) |\downarrow\rangle \) with \( \alpha(0) = \beta(0) = 1/\sqrt{2} \). Then, Schrödinger’s equation, \( i\hbar \dot{\chi} = \hat{H} \chi \) reads

\[ i\hbar [\dot{\alpha} |\uparrow\rangle + \dot{\beta} |\downarrow\rangle] = -\frac{\gamma B_0 \hbar}{2} \cos(\omega t) [\alpha |\uparrow\rangle - \beta |\downarrow\rangle]. \]  

(25.4.1)

This gives the two equations

\[ \dot{\alpha} = \frac{i\gamma B_0}{2} \cos(\omega t) \alpha, \quad \dot{\beta} = -\frac{i\gamma B_0}{2} \cos(\omega t) \beta, \]  

(25.4.2)

whose solutions with the correct initial conditions give

\[ \chi(t) = \frac{1}{\sqrt{2}} e^{i(\gamma B_0/2\omega) \sin(\omega t)} |\uparrow\rangle + \frac{1}{\sqrt{2}} e^{-i(\gamma B_0/2\omega) \sin(\omega t)} |\downarrow\rangle. \]  

(25.4.3)

(c) The \( z \) basis in terms of the \( x \) basis is

\[ |\uparrow\rangle = \frac{1}{\sqrt{2}} (|\rightarrow\rangle + |\leftarrow\rangle), \quad |\downarrow\rangle = \frac{1}{\sqrt{2}} (|\rightarrow\rangle - |\leftarrow\rangle). \]  

(25.4.4)

Hence, we may write \( \chi(t) \) as

\[ \chi(t) = \cos \left( \frac{\gamma B_0}{2\omega} \sin(\omega t) \right) |\rightarrow\rangle + i \sin \left( \frac{\gamma B_0}{2\omega} \sin(\omega t) \right) |\leftarrow\rangle. \]  

(25.4.5)

Therefore, the probability of measuring \( S_x = -\hbar/2 \) is

\[ P_{\chi}(x)(t) = \sin^2 \left( \frac{\gamma B_0}{2\omega} \sin(\omega t) \right). \]  

(25.4.6)

(d) A full spin flip occurs when \( P_{\chi}(x)(t) = 1 \) for some time \( t \). This requires the argument of \( \sin^2 \) in (25.4.6) to be \( \geq \pi/2 \), which requires \( \gamma B_0/2\omega \geq \pi/2 \), or \( B_0 \geq \pi \omega/\gamma \).
Chapter 25. QM in 3D

25.5 Hydrogen Ground State Calculations

(Spring 2007 Modern (Afternoon), Problem 2) Consider the electron in the ground state of a hydrogen atom.

(a) Determine the values for \( \langle r \rangle, \langle r^2 \rangle \) and the most probable value for the radius \( r \) of the orbit.

(b) Compute the probability that the electron is found inside the volume of the nucleus.

(c) Find the momentum-space wave function for the electron and determine the expectation value of the kinetic energy.

Solution:

(a) The only thing I seem to ever recall about the ground state of hydrogen is that its radial component goes like \( R_{10} \propto e^{-r/a_0} \) where \( a_0 \) is the Bohr radius. If we square this, multiply with \( r^2 \), and then integrate from \( r = 0 \) to \( r = \infty \), we must get 1. That fixes the coefficient:

\[
R_{10} = 2a_0^{-3/2}e^{-r/a_0}.
\]

Now, to find the expectation value of \( r \), simply do the same integral to find the normalization constant, but instead of multiplying with \( r^2 \), multiply with \( r^3 \). Similarly, multiply with \( r^4 \) for \( \langle r^2 \rangle \).

\[
\langle r \rangle = \frac{4}{a_0^3} \int_0^{\infty} r^3 e^{-2r/a_0} dr = \frac{3a_0}{2} \left[ \frac{4}{a_0^3} \int_0^{\infty} r^2 e^{-2r/a_0} dr \right] = \frac{3a_0^2}{2}, \tag{25.5.1}
\]

where the second equality follows from integration by parts and where the integral in the brackets is 1 by proper normalization of \( R_{10} \).

We also have

\[
\langle r^2 \rangle = \frac{4}{a_0^3} \int_0^{\infty} r^4 e^{-2r/a_0} dr = \frac{8}{a_0^3} \int_0^{\infty} r^3 e^{-2r/a_0} dr = 2a_0 \langle r \rangle = 3a_0^2, \tag{25.5.2}
\]

(b) Let \( r_0 \) be the characteristic radius of the hydrogen nucleus. After some tedious algebra, we find

\[
P(r < r_0) = \frac{4}{a_0^3} \int_0^{r_0} r^2 e^{-2r/a_0} dr = 1 - e^{-2x} [1 + 2x(1 + x)], \tag{25.5.3}
\]

where \( x = r_0/a_0 \ll 1 \). Expanding this, we find

\[
P(r < r_0) \approx \frac{4}{3} x^3 + O(x^4). \tag{25.5.4}
\]

Plugging in \( r_0 \approx 1.25 \times 10^{-15} \) m and \( a_0 \approx 5.29 \times 10^{-11} \) m, we find

\[
P(r < r_0) \approx 1.76 \times 10^{-14} \tag{25.5.5}
\]

(c) \( Y_0^0 = 1/2\sqrt{\pi} \) since the integral of its square over 4\( \pi \) solid angle is supposed to yield 1. Hence, the full ground state wavefunction is \( \psi_{100} = (\pi a_0^3)^{1/2}e^{-r/a_0} \). The Fourier transform is

\[
\psi(k) = \frac{1}{(2\pi)^{3/2}} \frac{1}{(2\pi a_0^3)^{1/2}} \int_0^{\infty} dr r^2 e^{-r/a_0} \int_{-1}^{1} e^{-ikr} dx \int_{0}^{2\pi} d\phi
\]

\[
= \frac{2}{\pi(2a_0)^{3/2} i k} \int_0^{\infty} dr r(e^{ikr} - e^{-ikr}) e^{-r/a_0}. \tag{25.5.6}
\]
Change variables to \( \xi = \frac{r}{a_0} (1 - ika_0) \) for the first and \( \xi = \frac{r}{a_0} (1 + ika_0) \) for the second integral:

\[
\psi(k) = \frac{2 \alpha^2}{\pi(2a_0)^{3/2} i k} \left[ \frac{1}{(1 - ika_0)^2} - \frac{1}{(1 + ika_0)^2} \right] \int_0^\infty \! d\xi \, \xi e^{-\xi} \\
= \frac{2 \alpha^2}{\pi(2a_0)^{3/2} i k} \frac{1}{1 + (ka_0)^2} \\
= \frac{(2a_0)^{3/2}}{\pi} \frac{1}{1 + (ka_0)^2}. \tag{25.5.7}
\]

The expectation value of the kinetic energy is

\[
- \langle E \rangle = \frac{\hbar^2}{2m} \int k^2 |\psi(k)|^2 \, d^3k = \frac{\hbar^2}{2m} \cdot 4\alpha \cdot \frac{(2a_0)^3}{\pi^2} \int_0^\infty \frac{k^4 \, dk}{[1 + (ka_0)^2]^4} \\
= \frac{8\hbar^2}{\pi ma_0^2} \int_{-\infty}^{\infty} x^4 \, dx \\
= \frac{4\hbar^2}{\pi ma_0} \int_{-\infty}^{\infty} x^2 \, dx \\
= \frac{\hbar^2}{2\pi ma_0^2} \int_{-\infty}^{\infty} (1 + x^2)^2 \\
= - \frac{\hbar^2}{2\pi ma_0^2} \int_{-\infty}^{\infty} x^2(1 + x^2) \\
= \frac{\hbar^2}{2\pi ma_0^2} \left[ \int_{-\infty}^{\infty} \frac{dx}{1 + x^2} - \int_{-\infty}^{\infty} \frac{dx}{x^2} \right] \\
= \frac{\hbar^2}{2ma_0}, \tag{25.5.8}
\]

where we extended the integral to \(-\infty\) and divided by 2 since the integrand is even, then the third, fourth and fifth lines follow from integration by parts, the sixth line from partial fractions expansion, and the final line from calculus of residues (or trigonometric substitution): \( \int_{-\infty}^{\infty} \frac{dx}{1 + x^2} = \int_{-\infty}^{\infty} \frac{dx}{(x+i)(x-i)} = 2\pi i \text{Res} \left( \frac{1}{(x+i)(x-i)}; i \right) = 2\pi i \frac{1}{2} = \pi \). This final expression is none other than \(-E_1\), the ground state energy of hydrogen, as expected.
25.6 Spherical Harmonic Oscillator

[Kevin G.] Calculate the energy spectrum of the three-dimensional isotropic harmonic oscillator in the rectilinear basis (quantum numbers \(n_x, n_y\) and \(n_z\)) and in the spherical basis (quantum numbers \(k, \ell\) and \(m\)).

**SOLUTION:**

The rectilinear case is easy since the problem factors into three independent and identical harmonic oscillators. Hence,

\[
E(n_x, n_y, n_z) = \hbar \omega \left(n_x + n_y + n_z + \frac{3}{2}\right).
\] (25.6.1)

The spherical case is more difficult. Since the potential is spherically symmetric, the wavefunction factors as \(\psi_{k\ell m}(r, \Omega) = R_{k\ell}(r) Y^m_\ell(\Omega)\). Recall that, if we define the reduced radial function \(u(r) \equiv r R(r)\), then \(u\) satisfies the Schrödinger equation with an effective potential:

\[
-\frac{\hbar^2}{2m} \frac{d^2 u}{dr^2} + \left[\frac{1}{2} m \omega^2 r^2 + \frac{\hbar^2}{2m} \ell(\ell + 1)\right] u = Eu.\] (25.6.2)

Define the following rescaled variables and constants:

\[
\kappa \equiv \frac{\sqrt{2mE}}{\hbar}, \quad \rho \equiv \kappa r, \quad \rho_0 \equiv \frac{\hbar \kappa^2}{m \omega}.\] (25.6.3)

Equation (25.6.2) may be written

\[
\left[\frac{d^2}{d\rho^2} + 1 - \left(\frac{\rho}{\rho_0}\right)^2 - \frac{\ell(\ell + 1)}{\rho^2}\right] u = 0.\] (25.6.4)

I have written it this way to closely resemble Eq. 4.56 (Griffiths p.146) for the hydrogen atom case. However, I will find it more convenient to define the new variable \(\xi \equiv \rho/\sqrt{\rho_0}\) in terms of which the equation reads

\[
\left[\frac{d^2}{d\xi^2} + \rho_0 - \xi^2 - \frac{\ell(\ell + 1)}{\xi^2}\right] u = 0.\] (25.6.5)

Consider the \(\xi \to 0\) limit:

\[
\left[\frac{d^2}{d\xi^2} - \frac{\ell(\ell + 1)}{\xi^2}\right] u \xrightarrow{\xi \to 0} 0.\] (25.6.6)

This is the same limit as in the case of hydrogen. The solution that does not blow up at \(\xi = 0\) is \(u(\xi) \sim \xi^{\ell+1}\). Now, consider the \(\xi \to \infty\) limit:

\[
\left[\frac{d^2}{d\xi^2} - \xi^2\right] u \xrightarrow{\xi \to \infty} 0.\] (25.6.7)

We cannot easily solve this equation. However, it does suggest peeling off a factor of \(e^{-\xi^2/2}\) since the part of the first term wherein the second derivative acts on the exponential cancels the second term altogether. Therefore, set

\[
u(\xi) = \xi^{\ell+1} e^{-\xi^2/2} v(\xi).\] (25.6.8)

After some algebra, one computes the second derivative:

\[
\frac{d^2 u}{d\xi^2} = \xi^\ell e^{-\xi^2/2} \left\{\xi \frac{d^2 v}{d\xi^2} + 2(\ell + 1 - \xi^2) \frac{dv}{d\xi} + \left[-(2\ell + 3)\xi + \xi^4 + \frac{\ell(\ell + 1)}{\xi}\right] v\right\}.\] (25.6.9)
25.6. SPHERICAL HARMONIC OSCILLATOR

Plugging this into (25.6.5) yields
\[ \xi \frac{d^2 v}{d \xi^2} + 2(\ell + 1 - \xi^2) \frac{dv}{d \xi} + (\rho_0 - 2\ell - 3)v = 0. \] (25.6.10)

Now, posit a MacLaurin expansion for \( v(\xi) \):
\[ v(\xi) = \sum_{j=0}^{\infty} c_j \xi^j = \sum_{j=1}^{\infty} c_{j-1} \rho^{j-1}. \] (25.6.11)

The following are useful expressions for the derivatives:
\[ \frac{dv}{d \xi} = \sum_{j=0}^{\infty} (j + 1)c_{j+1} \xi^j = \sum_{j=1}^{\infty} (j - 1)c_{j-1} \xi^{j-2}, \] (25.6.12)
\[ \frac{d^2 v}{d \xi^2} = \sum_{j=0}^{\infty} j(j + 1)c_{j+1} \xi^j. \] (25.6.13)

Plugging these into (25.6.10) gives
\[ 2(\ell + 1)c_1 + \sum_{j=1}^{\infty} [(j + 1)(j + 2\ell + 2)c_{j+1} - (2j + 2\ell + 1 - \rho_0)c_{j-1}] \xi^j = 0. \] (25.6.14)

It is clear that all the odd-numbered coefficients are linked via this equation and all the even-numbered coefficients are linked, but there are no odd-even links. Furthermore, \( c_1 = 0 \), which automatically implies \( c_3 = c_5 = \cdots = 0 \). Thus, we only have the even-numbered coefficients, which are linked via
\[ c_{j+1} = \frac{2j + 2\ell + 1 - \rho_0}{(j + 1)(j + 2\ell + 2)} c_{j-1}. \] (25.6.15)

Instead, we can write this as
\[ c_{j+2} = \frac{2j + 2\ell + 3 - \rho_0}{(j + 2)(j + 2\ell + 3)} c_j. \] (25.6.16)

Consider the large \( j \) limit:
\[ c_{j+2} \underset{j \to \infty}{\to} \frac{2}{j + 2} c_j. \] (25.6.17)

This implies \( c_{2j} = \frac{1}{j!} c_0 \). In this limit, we get
\[ v(\xi) \approx \sum_{j=0}^{\infty} c_{2j} \xi^{2j} = c_0 \sum_{j=0}^{\infty} \frac{(\xi^2)^j}{j!} = c_0 e^{\xi^2}, \] (25.6.18)

and so \( u(\xi) \) becomes
\[ u(\xi) \approx \xi^{\ell+1} e^{\xi^2/2}, \] (25.6.19)

which blows up as \( \xi \to \infty \).

As usual, the way out of this is to require the series for \( v \) to terminate. Looking at (25.6.16), this yields the condition
\[ \rho_0 = 2j_{\text{max}} + 2\ell + 3. \] (25.6.20)

Define the principal quantum number, \( k \equiv j_{\text{max}} + \ell + 1 \), so that
\[ E(k, \ell, m) = \frac{1}{2} \hbar \omega \rho_0 = \frac{1}{2} \hbar \omega (2k + 1) = \hbar \omega (k + \frac{1}{2}). \] (25.6.21)

We see that we can identify \( k \) with \( n + 1 \), where \( n = n_x + n_y + n_z \) and so we have the condition \( n = j_{\text{max}} + \ell \). Since the smallest possible value for \( j_{\text{max}} \) is zero and it jumps up in steps of 2, it follows that the allowed values of \( \ell \) are \( \ell = n, n - 2, \ldots (1 \text{ or } 0) \).
• Potential is a function of both coordinates. If it is time-independent, then we have the time-independent Schrödinger equation

$$\left[ -\hbar^2/2m_1 \nabla_1^2 - \hbar^2/2m_2 \nabla_2^2 + V(r_1, r_2) \right] \psi = E \psi.$$ 

• If $V$ is only a function of $r = r_1 - r_2$, then, transform to reduced mass and center of mass coordinates with $R = (m_1 r_1 + m_2 r_2)/(m_1 + m_2)$ the CoM position, and $\mu = m_1 m_2/(m_1 + m_2)$ the reduced mass. The CoM satisfies the free particle Schrödinger equation and the reduced mass the one with potential $V$ (c.f. Griffiths Problem 5.1, p.202).

• $(r_1, r_2) = \pm (r_2, r_1)$ where $+$ is for bosons and $-$ is for fermions. Note that $\psi$ may include spin states and so, for example, the singlet spin state for a pair of electrons is antisymmetric and must be accompanied by a symmetric spatial wavefunction, and the opposite scenario pertains to the triplet state.

• Helium: $\psi(r_1, r_2) = \psi_{n\ell m_\ell}(r_1) \chi_1(s_1) \psi_{n'\ell' m'(r_2)} \chi_2(s_2)$ properly anti-symmetrized, of course, if we neglect electron-electron interactions. Parahelium is spatially symmetric and Ortho-helium is spatially antisymmetric. [Note: since the charge of the nucleus is twice that of hydrogen, the Bohr radius is cut in half.]

• Periodic table: the $n^{th}$ shell can contain $2n^2$ electrons (2 spins and $n^2$ hydrogenic wavefunctions with a given value of $n$). Electron configurations: e.g. carbon ground state is $(1s)^2(2s)^2(2p)^{2}$. The order of filling is $1s, 2s, 2p, 3s, 3p, 4s, 3d, 4p, 5s, \ldots$.

• Spectroscopic notation: $2S+1L_J$ where $J, L, S$ are the total angular momentum, orbital, and spin.

• Hund’s rules: (1) All other things equal, the state with highest total spin $S$ will have the lowest energy. (2) For a given spin, the state with highest total orbital $L$, consistent with antisymmetrization, will have the lowest energy. (3) If a subshell $(n, \ell)$ is no more than half filled, then the lowest energy level has $J = |L - S|$; it it is more than half filled, then $J = L + S$ has the lowest energy.
26.1 Periodic Table, Pauli and Hund

[Fall 2001 Modern (Morning), Problem 3]

(a) What is the central field approximation? Explain why, to leading order, the possible states for a single electron can be labelled by the three integers \((n, \ell, m_\ell)\) and by \(m_s\), which is half integral, giving the significance of these four quantities.

(b) Draw a diagram of the first four periods of the Periodic Table, with a square representing each element. There is no need to name each element. How does quantum mechanics explain this pattern of elements? For a given shell, why do electrons first fill states of lowest \(\ell\)?

(c) That is the \(LS\) coupling scheme? In this scheme, what quantum numbers are used to describe the state of the atomic electrons? For carbon \((Z = 6)\) give

(a) the electron configuration,

(b) the possible terms which are allowed by the Pauli Principle. Which is the true ground state? (Use spectroscopic notation).

SOLUTION:

(a) Neglecting spin-orbit and spin-spin couplings, the Hamiltonian for \(Z\) electrons orbiting around a nucleus of charge \(Ze\) is

\[
H = \sum_{j=1}^{Z} \left[ -\frac{\hbar^2}{2m} \nabla_j^2 - \frac{\alpha\hbar Z}{r_j} \right] + \frac{\alpha\hbar}{2} \sum_{j \neq k}^{Z} \frac{1}{|r_j - r_k|}.
\]  

(26.1.1)

If we neglect electron-electron interactions, then each electron separately moves through the same Coulombic potential that is \(Z\) times as strong as that of hydrogen. Of course, the inner electrons screen the nucleus for the outer electrons. But the central field approximation maintains that whatever is this more complicated screened potential (plus whatever other interactions we neglected) can be approximated by a purely central potential (i.e. one that depends only on \(r_j\) for the \(j^{th}\) electron). Hence, each electron has a state \(\psi_{n\ell m}(r, \theta, \phi, s) = R_{n\ell}(r)Y_{\ell m}^{\ell}(\theta, \phi)\chi_{m_s}(s)\) where the radial equation for \(R_{n\ell}\) will be different from that of hydrogen. The relevant quantum numbers are \(n\) (labels solutions of radial equation), \(\ell, m_\ell\) (orbital angular momentum and its \(z\)-projection), \(m_s = \pm 1/2\) (\(z\)-projection of spin).

(b) There are \(2\sum_{\ell=0}^{n-1}(2\ell + 1) = 2n^2\) available electronic states per value of \(n\). There are two states available for the \(s\)-shell \((\ell = 0)\), then six states for the \(p\)-shell \((\ell = 1)\), then ten states for the \(d\)-shell \((\ell = 2)\). That is all we need for the first four rows.

Generically, the further out from the nucleus is an electron, the higher its energy (more positive) since the Coulombic potential energy goes as \(V \sim -1/r\) and since the inner electrons more effectively screen the nucleus. But, orbitals with higher values of \(\ell\) have larger values of mean radius. Thus, lower \(\ell\) values are filled first.

(c) This is spin-orbit coupling and, in addition to the quantum numbers above for each electron, we will need \(L\) and \(S\) or \(J = L + S\) and \(m_j\).

(i) \((1s)^2(2s)^2(2p)^2\).
(ii) Four electrons have \( \ell_j = 0 \) and two have \( \ell_j = 1 \) and thus the total orbital may be \( \ell = 0, 1, 2 \). The pairs of \( s \) electrons are in singlet spin states \((s_j = 0)\), whereas the last two may be in a singlet or a triplet and so \( s = 0, 1 \). The parity of the spherical harmonic is \((-1)\ell\) and of the singlet spin state is antisymmetric whereas the triplet state is symmetric. Thus, the total parity is \((-1)^{\ell+s+1}\), which is only \(-1\) for \((\ell, s) = (0, 0), (2, 0), (1, 1)\). The corresponding values of \( J \) are \(0, 2, \{0, 1, 2\}\), respectively. Hence, the allowed states are, in spectroscopic notation,

\[
\begin{align*}
1S_0, & \quad 1D_2, & \quad 3P_{0,1,2}.
\end{align*}
\]

We must use Hund’s rules to determine the true ground state. The first rule says that the ground state will be the one with the highest value of spin, which is one of \(3P_{0,1,2}\). The second rule does not distinguish between these states because they all have the same \( \ell \). Since the last subshell, \(2p\), has two electrons and is thus less than half filled, which would be three electrons, the third rule implies that the lowest energy state is the one with the smallest value of \( J \). Thus, the true ground state is \(3P_0\).
26.2 Hund’s Rules and the Aufbau Principle

[Kevin G.] Determine the term symbols for the ground states of the first ten elements describing the principles used.

**SOLUTION:**

H: $^2S_{1/2}$  $S = 1/2$ and $L = 0$. Subshell ≤ half-filled, so min $J = 1/2$.

He: $^1S_0$. Terms like $\uparrow\downarrow$ do not contribute. Also, filled subshells do not contribute. Hence, $S = L = J = 0$.

Li: $^2S_{1/2}$  Same as H.

Be: $^1S_0$  Same as He.

B: $^2P_{1/2}$  $S = 1/2$ and $L = 1$. Subshell ≤ half-filled, so min $J = 1/2$.

C: $^3P_0$. Aufbau: $\uparrow \uparrow \uparrow$ not $\uparrow \downarrow \downarrow$. Spin: $\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$. Max $S = 1$. This is symmetric, so need antisymmetric $L$ state. Orbital: $1 \otimes 1 = 0 \oplus 1 \oplus 2$. Highest is always fully symmetric: $|22\rangle = |1\rangle|1\rangle$ and its descendents, for example, $|21\rangle = \frac{1}{\sqrt{2}}(|1\rangle|0\rangle + |0\rangle|1\rangle)$. Orthogonally: $|11\rangle = \frac{1}{\sqrt{2}}(|1\rangle|0\rangle - |0\rangle|1\rangle)$, which is fully antisymmetric, along with descendents. Max antisymmetric $L = 1$. Subshell ≤ half-filled, so min $J = 0$.

N: $^4S_{3/2}$  Aufbau: $\uparrow \uparrow \uparrow \uparrow$ not $\uparrow \downarrow \downarrow \downarrow$. Max $S = 3/2$, which is symmetric, so need antisymmetric $L$ state. Orbital: $\otimes^31 = 0 \oplus 3(1) \oplus 2(2) \oplus 3$. $L = 3$ is symmetric and $L = 0$ is antisymmetric. $L = 1, 2$ have mixed symmetry. I believe that mixed symmetry will always prevail when the $L$ value is degenerate. Max antisymmetric $L = 0$. Subshell ≤ half-filled, so min $J = 3/2$.

O: $^3P_2$. Aufbau: $\uparrow \downarrow \uparrow \uparrow$ not $\uparrow \downarrow \uparrow \downarrow$. Only care about last two electrons. Same as C, except, since subshell > half-filled, max $J = 2$.

F: $^2P_{3/2}$  Only one last electron. Same as B, except max $J = 3/2$.

Ne: $^1S_0$. Filled.
26.3 Correction to Helium Ground State

[Kevin G. and Griffiths 5.11] If we neglect electron-electron interactions, then each electron orbiting the nucleus is in a hydrogenic wavefunction with appropriate changes to the Bohr radius and energies, etc.

(a) In this approximation, write down an expression for the wavefunction (including spin) for the ground state of helium and determine its energy.

(b) Experimentally, the energy is found to be $E_0 = -78.975$ eV, which you should find is very different from your result from part (a). This indicates that the approximation of neglecting electron-electron Coulombic repulsion is terrible. Let us try to include this effect: calculate $\langle |r_1 - r_2|^{-1} \rangle$ in your ground state from part (a).

(c) Use your result from part (b) to estimate the electron interaction energy in the ground state of helium. Express your answer in electron volts, and add it to the $E_0$ from part (a) to get a corrected estimate of the ground state energy. Compare this to the experimental value. (Of course, we’re still working with an approximate wavefunction, so don’t expect perfect agreement.)

SOLUTION:

(a) Each electron is in its own hydrogenic wavefunction $\psi_{nm}$ with a Bohr energy equal to $Z^2E_n$, which for helium is $4E_n$, where $E_n$ are the values for hydrogen. Since the energy does not depend on the spins of the electrons, we must find the spatial wavefunction with lowest energy, then the symmetry properties of that determines the those of the spin state. In this case, the lowest energy is achieved by $\psi_{00}(r_1, r_2) = \psi_{100}(r_1) \psi_{100}(r_2)$. We must be a bit careful in remembering the hydrogenic ground state. As before, $R_{10} \sim e^{-r/a_0}$ and if you square and multiply with $r^2$ and integrate, one must get 1, which fixes the normalization to $R_{10} = 2a_0^{-3/2}e^{-r/a_0}$. This is multiplied by $Y_0^0 = 1/\sqrt{\pi}$ and so the spatial wavefunction is $\psi_{100} = (\pi a_0^3)^{-1/2}e^{-r/a_0}$. However, the new Bohr radius is $a_0/Z = a_0/2$ and so, we actually have $\psi_{100} = (8/\pi a_0^3)^{1/2}e^{-2r/a_0}$. Since the ground state spatial wavefunction is symmetric, the spin state must be antisymmetric and thus the two electrons are in a singlet spin state:

$$\psi(r_1, r_2) = \frac{8}{\pi a_0^3} e^{-2(r_1 + r_2)/a_0} \frac{\langle |\uparrow\rangle - |\downarrow\rangle |\uparrow\rangle}{\sqrt{2}}\tag{26.3.1}$$

The energy is simply

$$E_0 = 4(E_1 + E_1) = 8(-13.6 \text{ eV}) = -109 \text{ eV}\tag{26.3.2}$$

(b) The spin component of the wavefunction is irrelevant. Set the $z$-axis of the $r_2$ integration to be along $r_1$ so that the polar angle is the angle between $r_1$ and $r_2$:

$$\langle \frac{1}{|r_1 - r_2|} \rangle = \left( \frac{8}{\pi a_0^3} \right)^2 \int d^3r_1 \int d^3r_2 \frac{e^{-4(r_1 + r_2)/a_0}}{[r_1^2 + r_2^2 - 2r_1r_2 \cos \theta]^{1/2}}$$

$$= \frac{128}{\pi a_0^6} \int d^3r_1 e^{-4r_1/a_0} \int_0^\infty dr_2 r_2^2 e^{-4r_2/a_0} \int_{-1}^{1} \frac{dx}{[r_1^2 + r_2^2 - 2r_1r_2 x]^{1/2}}$$

$$= \frac{2^7}{\pi a_0^6} \int d^3r_1 e^{-4r_1/a_0} \int_0^\infty dr_2 e^{-4r_2/a_0} \frac{r_2}{r_1} (r_1 + r_2 - |r_1 - r_2|).\tag{26.3.3}$$
We have \( \frac{1}{r_1 r_2} (r_1 + r_2 - |r_1 - r_2|) = \frac{1}{r_1} \) if \( r_1 > r_2 \) and \( \frac{1}{r_2} \) if \( r_1 < r_2 \). Thus,

\[
\left\langle \frac{1}{r_1 - r_2} \right\rangle = \frac{2^8}{\pi \alpha_0^2} \int d^3 r_1 e^{-4r_1/a_0} \left[ \frac{1}{r_1} \int_{r_1}^{r_2} dr_2 r_2^2 e^{-4r_2/a_0} + \int_{r_1}^{\infty} dr_2 r_2 e^{-4r_2/a_0} \right] \\
= \frac{2^8}{\pi \alpha_0^2} \frac{\alpha_0^5}{2^{10} 4\pi} \int_0^y dy \int_0^y dx x^2 e^{-x} + y \int_y^\infty dx x e^{-x},
\]

(26.3.4)

where \( x = 4r_2/a_0 \) and \( y = 4r_1/a_0 \). Let us compute each integral separately:

\[
\int_0^y dx x^2 e^{-x} = -y^2 e^{-y} + 2 \int_0^y dx xe^{-x} = 2 - [y(y + 2) + 2] e^{-y},
\]

(26.2.5a)

\[
y \int_y^\infty dx xe^{-x} = y^2 e^{-y} + \int_y^\infty dx e^{-x} = y(y + 1) e^{-y}.
\]

(26.2.5b)

Thus, the sum is

\[
\int_0^y dx x^2 e^{-x} + y \int_y^\infty dx xe^{-x} = 2 - (y + 2)e^{-y}.
\]

(26.3.6)

Now, we compute

\[
\int_0^\infty dy ye^{-y} = \frac{1}{4} \int_0^\infty dy ye^{-y} = \frac{1}{4} \int_0^\infty dy e^{-y} = \frac{1}{4},
\]

(26.2.7a)

\[
\int_0^\infty dy ye^{-y} = \int_0^\infty dy e^{-y} = 1,
\]

(26.2.7b)

\[
\int_0^\infty dy ye^{-2y} = \frac{1}{4} \int_0^\infty dy e^{-y} = \frac{1}{4}.
\]

(26.2.7c)

Thus, we have

\[
\left\langle \frac{1}{r_1 - r_2} \right\rangle = \frac{2}{\alpha_0} \int_0^\infty dy ye^{-y} - \frac{1}{\alpha_0} \int_0^\infty dy y(y + 2)e^{-2y} = \frac{5}{4\alpha_0}.
\]

(26.3.8)

(c) The interaction is the Coulomb one:

\[
V_{ee} \approx \left\langle \frac{\alpha \hbar c}{|r_1 - r_2|} \right\rangle = \frac{5\alpha \hbar c}{4\alpha_0} = -\frac{5}{2} E_1 = 34 \text{ eV}
\]

(26.3.9)

Adding this to the \(-109\) eV estimate of part (a) gives \( E_0 + V_{ee} = -75 \text{ eV} \), which is remarkably close to \(-79\) eV!
26.4 Nitrogen States

[Kevin G.] Write out the wavefunction for the ground state of Nitrogen, labelled as $|L, M_L, S, M_S\rangle$, in terms of the separate $m_\ell$ and $m_s$ states of the electrons. Here, $L$ is the total orbital angular momentum and $S$ is the total spin state. Determine the appropriate expansion for the state $|1, 1, \frac{1}{2}, \frac{1}{2}\rangle$, which is a $^2P$ state.

**SOLUTION:**

Hund’s first rule says maximize $S$. Nitrogen has three unpaired valence electrons. The possible total spin from three spin $\frac{1}{2}$ electrons can be $\frac{3}{2}$ (one of these) or $\frac{1}{2}$ (two of these). That is,

$$\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = 2\left(\frac{1}{2}\right) \oplus \frac{3}{2}.$$

(26.4.1)

The highest weight state is always totally symmetric (i.e. has an eigenvalue of 1 under every element of the permutation group on the three electrons). So, the $S = 3/2$ states are totally symmetric and can only be paired with totally antisymmetric $L$ states. The two $S = 1/2$ states have mixed symmetry, which we will explore momentarily.

For the ground state, Hund’s second rule tells us to find the maximum totally antisymmetric $L$ state. The three electrons are each in a $p$ state (or $\ell = 1$). The possible $L$ states are given by the decomposition

$$1 \otimes 1 \otimes 1 = 0 \oplus 3(1) \oplus 2(2) \oplus 3.$$  

(26.4.2)

We must work out which one of these is totally antisymmetric. Well, $L = 3$ is the highest weight and is thus totally symmetric. We will find that $L = 2$ and $L = 1$ have mixed symmetry and only $L = 0$ is totally antisymmetric. Therefore, the ground state has $L = 0$. Finally, the last subshell is $\leq$ half-filled and so we minimize $J$. The possible $J$ values for $L = 0$ and $S = 3/2$ is just $J = 3/2$. Therefore, the ground state is $^4S_{3/2}$.

Now, we must find its expansion. We must start all the way at the top (at $L = 3$) and work our way down to $L = 0$. We will expand a state $|L, M_L\rangle$ in terms of the product states $|m_\ell, m_s, m_\ell\rangle$. We start with $|3, 3\rangle = |1, 1, 1\rangle$ and keep lowering until we get to $|3, 0\rangle$. I will write the steps out just for the first lowering:

$$L_- |3, 3\rangle = \hbar\sqrt{3(3+1) - 3(3-1)} |3, 2\rangle = \hbar\sqrt{6} |3, 2\rangle,$$

$$(L_{1-} + L_{2-} + L_{3-}) |1, 1, 1\rangle = \hbar\sqrt{1(1+1) - 1(1-1)}(|1, 1, 0\rangle + |1, 0, 1\rangle + |0, 1, 1\rangle)$$

$$= \hbar\sqrt{2}(|1, 1, 0\rangle + |1, 0, 1\rangle + |0, 1, 1\rangle).$$

Setting these equal to each other yields the expansion for $|3, 2\rangle$. We can keep lowering thus getting the following expansions:

$$|3, 2\rangle = \frac{1}{\sqrt{3}}(|1, 1, 0\rangle + |1, 0, 1\rangle + |0, 1, 1\rangle),$$

(26.4.3a)

$$|3, 1\rangle = \frac{1}{\sqrt{12}}(2|1, 0, 0\rangle + 2|0, 1, 0\rangle + 2|0, 0, 1\rangle$$

$$+ |1, 1, -1\rangle + |1, -1, 1\rangle + |-1, 1, 1\rangle),$$

(26.4.3b)

$$|3, 0\rangle = \frac{1}{\sqrt{10}}(2|0, 0, 0\rangle + |1, 0, -1\rangle + |1, -1, 0\rangle + |-1, 1, 0\rangle$$

$$+ |0, 0, 1\rangle + |0, -1, 1\rangle + |0, 1, -1\rangle).$$

(26.4.3c)

Now, we must find two states built out of $|1, 1, 0\rangle$, $|1, 0, 1\rangle$ and $|0, 1, 1\rangle$ that are orthogonal to $|3, 2\rangle$ and to each other and are normalized. We can think of this as a change of basis from the
three product states to $|3, 2\rangle$ and two other states which we will call $|2, 2\rangle_a$ and $|2, 2\rangle_b$. We have a choice in the matter of how to write the latter two. One easy choice is

$$|2, 2\rangle_a = \frac{1}{\sqrt{2}}(-|1, 1, 0\rangle + |1, 0, 1\rangle),$$

$$|2, 2\rangle_b = \frac{1}{\sqrt{6}}(-|1, 1, 0\rangle - |1, 0, 1\rangle + 2|0, 1, 1\rangle).$$ \hspace{1cm} (26.4.4a, 26.4.4b)

The first choice was the easy one, but once it is made the second one is fixed. It should be clear that $|3, 2\rangle, |2, 2\rangle_a$ and $|2, 2\rangle_b$ form an orthonormal basis. We lower these again, noting that these have $L = 2$ rather than $L = 3$. We find

$$|2, 1\rangle_a = \frac{1}{2}(-|0, 1, 0\rangle + |0, 0, 1\rangle - |1, 1, -1\rangle + |1, -1, 1\rangle),$$

$$|2, 0\rangle_a = \frac{1}{2\sqrt{3}}(-|1, 0, -1\rangle + |1, -1, 0\rangle - |1, -1, 0\rangle + |0, 1, -1\rangle),$$

$$|2, 1\rangle_b = \frac{1}{2\sqrt{3}}(-2|1, 0, 0\rangle + |0, 1, 0\rangle + |0, 0, 1\rangle$$

$$- |1, 1, -1\rangle - |1, -1, 1\rangle + 2|1, -1, 1\rangle),$$

$$|2, 0\rangle_b = \frac{1}{2}(-|1, 0, -1\rangle - |1, -1, 0\rangle + |1, 1, 0\rangle + |1, 0, 1\rangle).$$ \hspace{1cm} (26.4.5a, 26.4.5b, 26.4.5c, 26.4.5d)

Now, we have to find three linear combinations of the states $|1, 0, 0\rangle, |0, 1, 0\rangle, |0, 0, 1\rangle, |1, 1, -1\rangle, |1, -1, 1\rangle$ and $|1, -1, 1\rangle$ that are orthonormal and orthogonal to $|3, 1\rangle, |2, 1\rangle_a$ and $|2, 1\rangle_b$. We will call these $|1, 1\rangle_a, |1, 1\rangle_b$ and $|1, 1\rangle_c$. What makes this task easier is we can consider the basis of product states to be divided into two sets of three: the three with two zeros and the three with two ones. We are clued in on this by the structure of $|3, 1\rangle$, particularly. When we dot product $|2, 1\rangle_a$ or $|2, 1\rangle_b$ with $|3, 1\rangle$, the result is 0 for each of the sets of three separately. So, we should be able to just switch the sign on the second set (the ones with two ones) in the states $|2, 1\rangle_a$ and $|2, 1\rangle_b$. The result will automatically be orthogonal to everything! The last state is fixed, but you can also just eyeball it:

$$|1, 1\rangle_a = \frac{1}{2}(-|0, 1, 0\rangle + |0, 0, 1\rangle + |1, 1, -1\rangle - |1, -1, 1\rangle),$$

$$|1, 1\rangle_b = \frac{1}{2\sqrt{3}}(-2|1, 0, 0\rangle + |0, 1, 0\rangle + |0, 0, 1\rangle$$

$$+ |1, 1, -1\rangle + |1, -1, 1\rangle - 2|1, -1, 1\rangle),$$

$$|1, 1\rangle_c = \frac{1}{\sqrt{15}}(|1, 0, 0\rangle + |0, 1, 0\rangle + |0, 0, 1\rangle$$

$$- 2|1, 1, -1\rangle - 2|1, -1, 1\rangle - 2|1, -1, 1\rangle).$$ \hspace{1cm} (26.4.6a, 26.4.6b, 26.4.6c)

We lower each of these to get

$$|1, 0\rangle_a = \frac{1}{2}(|1, 0, -1\rangle - |1, -1, 0\rangle - |1, -1, 0\rangle + |0, 0, 1\rangle),$$

$$|1, 0\rangle_b = \frac{1}{2\sqrt{3}}(|0, 1, 0\rangle - |1, -1, 0\rangle - |0, 1, 0\rangle$$

$$- |1, 0, 1\rangle + 2|0, -1, 1\rangle + 2|0, 1, -1\rangle),$$

$$|1, 0\rangle_c = \frac{1}{\sqrt{15}}(3|0, 0, 0\rangle - |1, 0, -1\rangle - |1, -1, 0\rangle - |0, 1, 0\rangle$$

$$- |1, 0, 1\rangle - |0, 1, 1\rangle - |0, 1, 1\rangle).$$ \hspace{1cm} (26.4.7a, 26.4.7b, 26.4.7c)

Now, we need to find a linear combination of the seven product states in Eqn. (26.4.7c) that is orthogonal to $|3, 0\rangle, |2, 0\rangle_{a,b}$ and $|1, 0\rangle_{a,b,c}$. There is only one such state (up to a minus sign, of course):

$$|0, 0\rangle = \frac{1}{\sqrt{6}}(|1, 0, -1\rangle - |1, -1, 0\rangle + |1, -1, 0\rangle - |0, 0, 1\rangle + |0, -1, 1\rangle - |0, 1, -1\rangle).$$ \hspace{1cm} (26.4.8)
As promised, this state is totally antisymmetric. For example, if we switch electrons 1 and 2, \(|1, 0, -1\rangle\) turns into \(|0, 1, -1\rangle\) and vice versa. These two terms have opposite signs in Eqn. \((26.4.8)\), so it works out!

Now, we must do the same for the spin degree of freedom. Thankfully, this is essentially the same:

\[
|\frac{3}{2}, \frac{3}{2}\rangle = |\uparrow\uparrow\uparrow\rangle, \tag{26.4.9a}
\]

\[
|\frac{3}{2}, \frac{1}{2}\rangle = |\frac{1}{2}\rangle|\uparrow\downarrow\downarrow\rangle + |\frac{1}{2}\rangle|\downarrow\uparrow\downarrow\rangle + |\frac{1}{2}\rangle|\downarrow\downarrow\uparrow\rangle, \tag{26.4.9b}
\]

\[
|\frac{1}{2}, \frac{3}{2}\rangle_a = |\frac{1}{2}\rangle|\uparrow\downarrow\uparrow\rangle + |\frac{1}{2}\rangle|\downarrow\uparrow\downarrow\rangle, \tag{26.4.9c}
\]

\[
|\frac{1}{2}, \frac{1}{2}\rangle_b = |\frac{1}{2}\rangle|\downarrow\uparrow\downarrow\rangle - |\frac{1}{2}\rangle|\uparrow\downarrow\downarrow\rangle + \frac{2}{3}|\downarrow\downarrow\downarrow\rangle. \tag{26.4.9d}
\]

\(S = 3/2\) is totally symmetric and the two \(S = 1/2\) states have mixed symmetry.

There are four ground states, \(|0, 0, \frac{3}{2}, \pm \frac{1}{2}\rangle\) and \(|0, 0, \frac{3}{2}, \pm \frac{1}{2}\rangle\) simply given by the product of Eqn. \((26.4.8)\) with Eqn. \((26.4.9a)\), Eqn. \((26.4.9b)\) and their descendents, which look identical, just with more down arrows (e.g. \(|\frac{3}{2}, -\frac{1}{2}\rangle = |\downarrow\downarrow\rangle\)).

Now, we turn our attention to the \(|1, 1\rangle\) states. Notice that \(|1, 1\rangle_a\) is totally symmetric, just like all of the \(L = 3\) states. Since there is no totally antisymmetric total spin, these states cannot exist! We only have the two, \(|1, 1\rangle_a\) and \(|1, 1\rangle_b\), with which to work. We must join some linear combination of these two with some linear combination of the two \(S = 1/2\) states to produce a totally antisymmetric state.

Under a switch of electrons 2 and 3, \(|1, 1\rangle_a \rightarrow -|1, 1\rangle_a\) while \(|1, 1\rangle_b \rightarrow |1, 1\rangle_b\) and \(|\frac{1}{2}, \frac{1}{2}\rangle_a \rightarrow -|\frac{1}{2}, \frac{1}{2}\rangle_a\) while \(|\frac{1}{2}, \frac{1}{2}\rangle_b \rightarrow |\frac{1}{2}, \frac{1}{2}\rangle_b\). Thus, the most general state is

\[
|1, 1, \frac{1}{2}, \frac{1}{2}\rangle = \alpha |1, 1\rangle_a |\frac{1}{2}, \frac{1}{2}\rangle_b + \beta |1, 1\rangle_b |\frac{1}{2}, \frac{1}{2}\rangle_a. \tag{26.4.10}
\]

Ensuring that this is antisymmetric under a switch of electrons 1 and 2 is more difficult since neither \(|1, 1\rangle_a\) nor \(|1, 1\rangle_b\) is an eigenstate of this operator. Instead, each one turns into a linear combination of the two. In other words, they form a two-dimensional representation of the permutation group on three electrons. The same goes for the spin. Straightforward calculation yields

\[
(12)|1, 1\rangle_a = \frac{1}{2}\left(-|1, 0, 0\rangle + |0, 0, 1\rangle + |1, 1, -1\rangle - |1, 1, 1\rangle\right) = \frac{1}{2} |1, 1\rangle_a + \frac{\sqrt{2}}{2} |1, 1\rangle_b, \tag{26.4.11a}
\]

\[
(12)|1, 1\rangle_b = \frac{1}{2\sqrt{2}}\left(-|0, 1, 0\rangle + |1, 0, 0\rangle + |0, 0, 1\rangle + |1, 1, -1\rangle - |1, 1, 1\rangle - 2|1, -1, 1\rangle\right) = \sqrt{\frac{2}{3}} |1, 1\rangle_a - \frac{1}{2} |1, 1\rangle_b. \tag{26.4.11b}
\]

Therefore, we have

\[
(12)|1, 1, \frac{1}{2}, \frac{1}{2}\rangle = \alpha |1, 1\rangle_a + \frac{\sqrt{2}}{2} |1, 1\rangle_b \left(\frac{\sqrt{2}}{2} |\frac{1}{2}, \frac{1}{2}\rangle_a - \frac{1}{2} |\frac{1}{2}, \frac{1}{2}\rangle_b\right)
\]

\[
\beta \left(\frac{\sqrt{2}}{2} |1, 1\rangle_a - \frac{1}{2} |1, 1\rangle_b \right) \left(\frac{1}{2} |\frac{1}{2}, \frac{1}{2}\rangle_a + \frac{\sqrt{2}}{2} |\frac{1}{2}, \frac{1}{2}\rangle_b\right)
\]

\[
= \frac{\sqrt{2}}{2}(\alpha + \beta) |1, 1\rangle_a |\frac{1}{2}, \frac{1}{2}\rangle_a - |1, 1\rangle_b |\frac{1}{2}, \frac{1}{2}\rangle_b
\]

\[
- \frac{1}{2}(\alpha - 3\beta) |1, 1\rangle_a |\frac{1}{2}, \frac{1}{2}\rangle_a + \frac{1}{2}(3\alpha - \beta) |1, 1\rangle_b |\frac{1}{2}, \frac{1}{2}\rangle_a. \tag{26.4.12}
\]

Setting this equal to \(-|1, 1, \frac{1}{2}, \frac{1}{2}\rangle\) requires \(\beta = -\alpha\) and then normalization gives \(\alpha = \pm \sqrt{\frac{1}{2}}\).

Therefore,

\[
|1, 1, \frac{1}{2}, \frac{1}{2}\rangle = \frac{1}{\sqrt{2}} \left(|1, 1\rangle_a |\frac{1}{2}, \frac{1}{2}\rangle_b - |1, 1\rangle_b |\frac{1}{2}, \frac{1}{2}\rangle_a\right). \tag{26.4.13}
\]

At this point, you can just plug everything in and simplify. I won’t bother. The same procedure will work for the \(L = 2\) state, \(^2D\).
Chapter 27

Time-Independent Perturbation Theory

• Let $H = H^0 + \lambda H'$ and expand $\psi_n = \psi_n^0 + \lambda \psi_n^1 + \cdots$ and $E_n = E_n^0 + \lambda E_n^1 + \cdots$. Assume that $\psi_n^0$ are properly normalized.

• Define $V_{mn} = \langle \psi_m^0 | H' | \psi_n^0 \rangle$ and $\Delta_{mn} = E_m^0 - E_n^0$.

• First order energy correction: $E_n^1 = V_{nn}$.

• First order wavefunction correction: $\psi_n^1 = \sum_{m \neq n} \frac{V_{mn}}{\Delta_{nm}} \psi_m^0$.

• Second order energy correction: $E_n^2 = \sum_{m \neq n} |V_{nm}|^2 / \Delta_{nm}$.

• If there is a degenerate set of zeroth order states $\psi_a^0$ where $a = 1, \ldots, N$ and $N$ is the degeneracy, then you must diagonalize the perturbation, $H'$, within this subspace before continuing on with the usual program outlined above. Where the denominator in $E_2$ or $\psi^2$ vanishes, the numerator now also vanishes and all is well.

• Orders of corrections to Bohr energies (which are $\alpha^2$ in units of $m_ec^2$): (1) Fine structure ($\alpha^4$); (2) Lamb ($\alpha^5$); (3) Hyperfine ($(m_e/m_p)\alpha^4$).

• Fine structure: $E_{ls}^1 = \left[ \frac{\alpha^2}{\pi^n} \left( \frac{3}{4} - \frac{n}{j+1/2} \right) \right] E_n$, breaks the degeneracy in $\ell$, but preserves degeneracy in $j$. The quantum numbers $m_\ell$ and $m_s$ are no longer good (since $L, S$ do not commute with $H_{so}^j \sim L \cdot S$. The good quantum numbers are $n, \ell, s, j$ and $m_j$.

• Zeeman effect: $H'_{Z} = \frac{e}{2m}(L + 2S) \cdot B_{ext}$. Weak field: fine structure dominates and its good quantum numbers hold $(n, \ell, j, m_j)$, then $E_{ls}^2 = \mu_B g_J B_{ext} m_j$ where $g_J = 1 + j(j+1)-l(l+1)+3/4$ is the Landé g-factor. Strong field: fine structure is subdominant and good quantum numbers are $n, \ell, m_\ell$ and $m_s$, NOT $j$ and $m_j$.

• Hyperfine: electron moves in magnetic field of magnetic dipole moment of proton. This magnetic field has (1) the regular dipole part; and (2) a delta function part. (1) affects only $\ell \neq 0$ states, whereas (2) affects only $\ell = 0$ states. The effect of (1) is more complicated and not in Griffiths, I don’t think. The effect on $\ell = 0$ states is $E_{hl}^1 = \frac{2g_p h^4}{3m_e m_p^2 c^2 a^3} ((S/h)^2 - \frac{3}{2})$, where $g_p$ is the proton gyromagnetic ratio, which is $g_p \approx 5.6$, and $a \approx 0.5$ Å is the Bohr radius. The separate spins are no longer good quantum numbers and must be replaced with the total spin $S = S_p + S_e$ and its projection.
27.1 K Meson-Proton Bound State Transition

[Fall 1987 Modern (Afternoon), Problem 6] A $K^-$ meson ($M_{K^-} = 494$ MeV) and a proton can form a bound atomic state. Take them both to be point particles subject only to the Coulomb force.

(a) Calculate the $2P$ to $1S$ transition energy.

(b) Estimate the probability of finding the $K^-$ and $p$ within $10^{-13}$ cm of each other when they are in the $1S$ state.

(c) Suppose that the strong interaction between the $K^-$ and $p$ can be represented by an attractive square well potential of 50 MeV and range $10^{-13}$ cm. Using lowest order perturbation theory, estimate the change in the transition energy calculated in (a) due to the strong interaction.

**SOLUTION:**

(a) The reduced mass is $\mu = \frac{(494)(938)}{494+938} = 323$ MeV = 646$m_e$. Let $\zeta = 646$ measure how much greater the reduced mass is compared to the electron mass. The energy states in this case are $E_n' = -\alpha^2\mu e^2/2n^2 = \zeta E_n$, where $E_n$ are the usual hydrogen energy levels. For a $2P \rightarrow 1S$ transition in hydrogen, the transition energy is $\Delta E = -\frac{13\alpha}{4} - (-13.6) = 10.2$ eV. Thus, for the $K^-p$ system, $\Delta E' = \zeta \Delta E = 6.6$ keV.

(b) The new Bohr radius is $a = \hbar/\alpha\mu c = a_0/\zeta \approx 77$ fm, given that $a_0 \approx 0.529 \text{ Å}$. The wavefunction of the system in the variable $r$ denoting the displacement between the two particles is just the ground state hydrogenic wavefunction with the new value of the Bohr radius: $\psi_{100}(r) = (\pi a_0^3)^{-1/2}e^{-r/a}$. The probability of finding the two particles within a distance $r_0$ of each other is

$$P_{1S}(r < r_0) = \frac{4}{a^3} \int_0^{r_0} dr r^2 e^{-2r/a} = 1 - e^{-2x} [1 + 2x(1 + x)] \approx \frac{4x^3}{3},$$

(27.1.1)

where $x = r_0/a << 1$ (c.f. Problem 25.5 part (b)). In our case, $x = r_0/a = 1/77$:

$$P_{1S}(r < r_0) \approx \frac{4(1/77)^3}{3} \approx 3 \times 10^{-6}$$

(27.1.2)

Incidentally, you get the same result in a much simpler fashion by simply approximating $e^{-2r/a}$ by 1 since $r_0 << a$ and so $e^{-2r/a}$ is essentially constant over the integration region.

We will follow this prescription below.

(c) Set $H' = -50$ MeV for $r < r_0 = 1$ fm and 0 outside. The change to the $1S$ energy is simple:

$$\Delta E_{1S} = \langle H' \rangle_{1S} = (-50 \text{ MeV})P_{1S}(r < r_0) = -150 \text{ eV.}$$

We will also need the probability that the two particles are within $r_0$ of each other when they are in the $2P$ state. For this, we would need to know the $2P$ state! It’s a good idea to know a few things at least about these states: $R_{n\ell} \sim e^{-x/n}x^{\ell}f_{n-\ell-1}(x)$, where $x = r/a$ and $a$ is the appropriate Bohr radius, and $f_{n-\ell-1}(x)$ is some degree $n - \ell - 1$ polynomial in $x$. For the $R_{n,n-1}$ states, this gives the full state up to a normalization constant, which can be computed. Doing so yields $R_{21}(r) = (2a_0^3)^{-1/2}(r/a)e^{-r/2a}$. Hence

$$P_{2P}(r < r_0) = \frac{1}{24a_0^3} \int_0^{r_0} dr r^2 (r/a)^2 e^{-r/a} \approx \frac{1}{24} \int_0^{r_0/a} dx x^4 = \frac{(r_0/a)^5}{120} = 3.1 \times 10^{-12}.$$  

(27.1.3)
The change to the $2P$ energy is $\Delta E_{2P} = (-50 \text{ MeV})P_{2P}(r < r_0) = -150 \text{ } \mu\text{eV}$, which is negligible. Thus,

$$\Delta E(2P \rightarrow 1S) \approx 150 \text{ eV}$$  \hspace{1cm} (27.1.4)

Since $E_{1S}$ is lowered and $E_{2P}$ is hardly changed, the transition energy should indeed increase.
27.2 Hyperfine and Weak Zeeman Effects

[Fall 2003 Modern (Morning), Problem 5] A cesium atom is in its ground electronic state $6S_{1/2}$. Its nuclear spin is $7/2$.

(a) Given that the hyperfine-splitting Hamiltonian has the form $H_{\text{hfs}} = a \mathbf{I} \cdot \mathbf{J}$, where $\mathbf{I}$ is the nuclear spin vector operator, and $\mathbf{J}$ is the total electron angular momentum operator, what is the hyperfine splitting of the ground electronic state of the cesium atom in the absence of a magnetic field?

(b) In the presence of a small magnetic field, what would be the Zeeman-split energy levels of this atom, up to a proportionality constant? Sketch these energy levels versus the magnetic field.

SOLUTION:

(a) Define the total angular momentum operator $\mathbf{F} = \mathbf{I} + \mathbf{J}$ in terms of which $H_{\text{hfs}} = a \mathbf{I} \cdot \mathbf{J} = \frac{a}{2} \hat{F}^2 - \hat{I}^2 - \hat{J}^2$, or

$$\langle H_{\text{hfs}} \rangle = \frac{\hbar^2 a}{2} \left[ F(F+1) - I(I+1) - J(J+1) \right]. \quad (27.2.1)$$

In our case, $J = 1/2$ and $I = 7/2$ and so $F = \{ \frac{7}{2} - \frac{1}{2}, \frac{7}{2} + \frac{1}{2} \} = \{ 3, 4 \}$. Thus,

$$\langle H_{\text{hfs}} \rangle_{F=3} = \frac{9\hbar^2 a}{8}, \quad \langle H_{\text{hfs}} \rangle_{F=4} = -\frac{9\hbar^2 a}{4}. \quad (27.2.2)$$

Therefore, the two states are split by $\Delta E_{\text{hfs}} = 4\hbar^2 a$.

(b) The Zeeman Hamiltonian is $H_Z = -(\mu_e + \mu_n) \cdot \mathbf{B} \approx -\mu_e \cdot \mathbf{B} = -\gamma \mathbf{J} \cdot \mathbf{B}$, where $\mu_n$ is the nuclear magneton and is much smaller than the electron magneton since it is inversely proportional to mass.

The conserved quantity in this problem is $\mathbf{F}$, not $\mathbf{I}$ or $\mathbf{J}$ separately. Hence, the latter two precess rapidly around their sum [c.f. Griffiths Figure 6.10, p.278, with the changes $(\mathbf{L}, \mathbf{S}, \mathbf{J}) \rightarrow (\mathbf{I}, \mathbf{J}, \mathbf{F})]$. Therefore, the time average value of $\mathbf{J}$ is just its projection along $\mathbf{F}$:

$$\langle \mathbf{J} \rangle = \frac{\mathbf{J} \cdot \mathbf{F}}{\hat{F}^2} \mathbf{F}. \quad (27.2.3)$$

Now, $\mathbf{I} = \mathbf{F} - \mathbf{J}$ and so $\hat{I}^2 = \hat{F}^2 + \hat{J}^2 - 2\mathbf{F} \cdot \mathbf{J}$, or $\mathbf{J} \cdot \mathbf{F} = \frac{1}{2} \hat{F}^2 + \hat{J}^2 - \hat{I}^2$, and so

$$\langle \mathbf{J} \rangle = \frac{F(F+1) + J(J+1) - I(I+1)}{2F(F+1)} \langle \mathbf{F} \rangle. \quad (27.2.4)$$

Align the z-axis with the magnetic field so that $\langle \mathbf{F} \rangle \cdot \mathbf{B} = \hbar m_F B$. Then, finally,

$$\langle H_Z \rangle = -\gamma m_F B \frac{F(F+1) + J(J+1) - I(I+1)}{2F(F+1)}. \quad (27.2.5)$$

In the two states $F = 3, 4$, this gives

$$\langle H_Z \rangle_{3,4} = \pm \frac{1}{8} \gamma m_F B, \quad (27.2.6)$$

where the plus sign is for $F = 3$ and the minus for $F = 4$. This is just linear in $B$, so I won’t bother to draw it. This splitting is in the opposite direction to the splitting due to hyperfine. Thus, I would say $\Delta E_Z = -\frac{1}{4} \gamma m_F B$. An external magnetic field tends to reduce the the original hyperfine splitting.
27.3 Perturbed 2D Harmonic Oscillator

Consider a quantum system described by the Hamiltonian $H = H_0 + H'$, where $H_0 = \frac{1}{2m}(p_x^2 + p_y^2) + \frac{1}{2}m\omega^2(x^2 + y^2)$ is the two-dimensional harmonic oscillator Hamiltonian, and where the perturbing Hamiltonian is given by $H' = Kxy$. Here, $K$ is a constant. Evaluate the first and second order energy and first order wavefunction corrections to the states whose unperturbed energy (i.e. energy if $K$ were zero) is $2\hbar\omega$. \[\text{[Hint: Think parity and raising and lowering operators.]}\]

The following are some useful formulas

\[
\int_0^\infty dx x^n e^{-ax} = n!/a^{n+1} \quad (a > 0), \quad \int_{-\infty}^\infty e^{-x^2} = \sqrt{\pi}.
\]

For the one-dimensional simple harmonic oscillator,

\[
x = \sqrt{\hbar/2m}\omega (a^\dagger + a), \quad p_x = i\sqrt{\hbar m\omega/2}(a^\dagger - a), \quad a^\dagger |n\rangle = \sqrt{n + 1} |n + 1\rangle, \quad a |n\rangle = \sqrt{n} |n - 1\rangle.
\]

**SOLUTION:**

The unperturbed states are product states $|n_xn_y\rangle$ with energies $E = \hbar\omega(n_x + n_y + 1)$. There are two states $|01\rangle$ and $|10\rangle$ both have energy $2\hbar\omega$ (and are the only such ones). In the subspace spanned by these two states, $H'$ is not diagonal. Define $\epsilon \equiv \hbar K/2m\omega$, in terms of which the perturbation may be written

\[
H' = \epsilon(a_{x^\dagger} + a_x)(a_{y^\dagger} + a_y).
\]

Define $|1\rangle \equiv |01\rangle$ and $|2\rangle \equiv |10\rangle$, and $M_{ij} = \langle i|H'|j\rangle$. Then, it is easy to see that the matrix $M$ is $M = \epsilon \sigma_x$ where $\sigma_x$ is the standard first Pauli matrix. The eigenvectors and eigenvalues are

\[
|\pm\rangle = \frac{1}{\sqrt{2}}(|1\rangle \pm |2\rangle), \quad \text{eigenvalue } \pm \epsilon.
\]

Hence, the first order energy corrections are

\[
E_{\pm} = 2\hbar\omega \pm \frac{\hbar K}{2m\omega}.
\]

Define $\epsilon = K/2m\omega^2$ so that the above reads

\[
E_{\pm} = E^{(0)} + E^{(1)} = (2 \pm \epsilon)\hbar\omega.
\]

The first order wavefunction correction is

\[
|\pm\rangle^{(1)} = \sum_{n_xn_y} \frac{|n_xn_y\rangle \langle n_xn_y|H'|\pm\rangle^{(0)}}{E_{\pm}^{(0)} - E(n_x, n_y)^{(0)}}.
\]

Only $|21\rangle$ and $|12\rangle$ can be connected $|\pm\rangle$ to this order:

\[
\langle 12|H'|01\rangle = \sqrt{2} \epsilon \hbar\omega, \quad \langle 21|H'|10\rangle = \sqrt{2} \epsilon \hbar\omega.
\]

This implies

\[
\langle 12|H'|\pm\rangle^{(0)} = \epsilon \hbar\omega, \quad \langle 21|H'|\pm\rangle^{(0)} = \pm \epsilon \hbar\omega.
\]
Therefore, we get

\[ |\pm\rangle = |\pm\rangle^{(0)} + \frac{|12\rangle \epsilon \hbar \omega}{2\hbar \omega - 3\hbar \omega} + \frac{|21\rangle (\pm \epsilon \hbar \omega)}{2\hbar \omega - 3\hbar \omega} = |\pm\rangle^{(0)} - \epsilon(|12\rangle \mp |21\rangle). \quad (27.3.8) \]

The second order energy correction is

\[
E^{(2)} = \sum_{n_x, n_y \neq 1} \frac{|\langle n_x n_y | H' | \pm \rangle^{(0)}|^2}{E_{\pm}^{(0)} - E(n_x, n_y)^{(0)}} \\
= \frac{|\epsilon \hbar \omega|^2 + |\pm \epsilon \hbar \omega|^2}{2\hbar \omega - 3\hbar \omega} \\
= -2\epsilon^2 \hbar \omega. \quad (27.3.9)
\]

Thus, \[ E_{\pm} = (2 \pm \epsilon - 2\epsilon^2) \hbar \omega. \]
27.4 Perturbed Spin-1/2 Harmonic Oscillator

[Spring 2009 Modern (Morning), Problem 5] A spin-1/2 particle is confined to a one-dimensional harmonic potential, governed by the Hamiltonian \( H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 + \lambda x \sigma_z \), where \( \sigma_z \) is a spin Pauli matrix.

(a) Find the energy eigenvalues and the wavefunction of the ground state(s) for this Hamiltonian. Now, an additional identical particle is added. The particles are fermions and can be taken to be non-interacting. What is the ground state energy of this two-particle system?

(b) A weak perturbation, \( \Delta H = \epsilon \sigma_x \), is added to the Hamiltonian above. Returning to the problem of a single particle, calculate the correction to the ground state energy to first order in \( \epsilon \). Write down the wavefunction of the ground state to lowest order, such that \( \langle \psi \rvert \Delta H \lvert \psi \rangle \) gives the energy correction obtained above.

SOLUTION:

(a) In matrix form, the Hamiltonian reads

\[
H = \begin{pmatrix}
\frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 + \lambda x & 0 \\
0 & \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 - \lambda x
\end{pmatrix}.
\]

After completing the squares separately to \( x_\pm = x \pm \frac{\lambda}{m\omega} \),

\[
H = \begin{pmatrix}
\frac{p^2}{2m} + \frac{1}{2}m\omega^2 x_+^2 - \frac{\lambda^2}{2m\omega^2} & 0 \\
0 & \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x_-^2 - \frac{\lambda^2}{2m\omega^2}
\end{pmatrix}.
\]

Hence, the system is equivalent to two displaced harmonic oscillators with exactly the same spectrum (one for spin up and one for spin down):

\[
\langle E_n \rangle_\pm = (n + \frac{1}{2})\hbar \omega - \frac{\lambda^2}{2m\omega^2}. \tag{27.4.3}
\]

Let us calculate the ground state of the usual SHO. The lowering operator is \( a = i \sqrt{\frac{p^2/2m}{\hbar \omega}} + \sqrt{\frac{m\omega^2 x^2/2}{\hbar^2}} \) (or \( a = \sqrt{\frac{m\omega^2}{2\hbar}} \) and \( \frac{d}{dx} = \frac{\hbar}{\sqrt{m\omega}} \)). Define \( \xi = \sqrt{\frac{m\omega}{2\hbar}} x \), then, \( a = \frac{d}{d\xi} + \xi \). The ground state is defined by \( a \psi_0 = 0 \), or \( \frac{d\psi_0}{d\xi} = -\xi \psi_0 \). This is easily solved by \( \psi_0 = Ae^{-\xi^2} \).

Proper normalization implies

\[
\psi_0(x) = \left( \frac{m\omega}{\pi \hbar} \right)^{1/4} e^{-\frac{m\omega}{2\hbar} x^2}. \tag{27.4.4}
\]

The two ground states in our present case are

\[
\lvert \psi_0 \rangle_\pm = \psi_0(x_\pm) \lvert \pm \rangle, \tag{27.4.5}
\]

where \( \lvert + \rangle = \lvert \uparrow \rangle \) and \( \lvert - \rangle = \lvert \downarrow \rangle \).

(b) The ground state is degenerate and thus we must use degenerate perturbation theory. Define \( \lvert n \rangle_\pm \) such that \( \langle x \rvert 0 \rangle_\pm = \psi_0(x_\pm) \). Then, \( \lvert \psi_0 \rangle_\pm = \lvert 0 \rangle_\pm \lvert \pm \rangle \). The perturbing operator is \( \Delta H = \epsilon \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \). The diagonal terms vanish: \( \langle \psi_0 \rvert \Delta H \lvert \psi_0 \rangle_\pm = 0 \). The off-diagonal terms are
\[ \pm \langle \psi_0 | \Delta H | \psi_0 \rangle = \epsilon_{\pm} (0|0)_{\pm}. \]

**Method 1:**

\[ \pm (0|0)_{\pm} = \sqrt{\frac{m\omega}{\pi\hbar}} \int e^{-\frac{m\omega}{2\hbar}(x^2 + y^2)} dx = \sqrt{\frac{m\omega}{\pi\hbar}} \int \exp\left\{ -\frac{m\omega}{2\hbar} \left[ 2x^2 + \frac{2\lambda^2}{m^2\omega^2} \right] \right\}. \]

We recognize that this is simply \( \pm (0|0)_{\pm} = e^{-\lambda^2/m\hbar^3} (0|0) = e^{-\lambda^2/m\hbar^3}. \)

Hence, the perturbation expectation matrix is

\[ \langle \Delta H \rangle = \epsilon e^{-\lambda^2/m\hbar^3} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \]  

(27.4.6)

Diagonalizing simply turns \( \sigma_x \) into \( \sigma_z \). Thus,

\[ (E_0)_{\pm} = \frac{1}{2} \hbar \omega - \frac{\lambda^2}{2m\omega^2} \pm \epsilon e^{-\lambda^2/m\hbar^3} \]  

(27.4.7)

The true ground state is the lower energy whose corresponding wavefunction is

\[ \frac{1}{\sqrt{2}} (|\psi_0\rangle_+ - |\psi_0\rangle_-). \]

**Method 2:** Relate raising and lowering operators in \( x \) and \( x_\pm \).

**Method 3:** Use Zassenhaus as in Problem 29.5.
27.5 Positronium in a Magnetic Field

[Based on Spring 2009 Modern (Afternoon), Problem 6] Positronium is an atom made up of an electron and a positron (electron anti-particle).

(a) As a “zeroth” order approximation, neglect everything (e.g. fine structure, hyperfine structure, etc.) except the Coulombic interaction between the electron and positron. Relate the “zeroth” order positronium energies to those of hydrogen.

(b) Taking spin into account, each of the levels of part (a) are 4-degenerate (since the electron and positron are both spin-1/2 particles). Taking spin-spin interactions into account, $H = H_0 - A \mu_p \cdot \mu_e$, where $\mu$ is the magnetic moment, $p$ stands for positron and $e$ for electron. Place the positronium in a uniform and constant external magnetic field, $B$. What effect do these things have on the energy of a level?

**SOLUTION:**

(a) The Bohr Hydrogen energies are $E_n = -\frac{4 \pi \alpha^2 m_e e^2}{n^2}$, where $\alpha = e^2/4\pi \varepsilon_0 hc \approx 1/137$ is the fine structure constant. To correct for finite proton mass, one uses the reduced electron mass $\mu = \frac{m_e m_p}{m_e + m_p}$. In this case, $m_p = m_e$, so $\mu = m_e/2$. Thus, Positronium energies are half of Bohr energies

(b) $\mu = \gamma S$, where $\gamma$ is gyromagnetic ratio. $\gamma = -g \mu_B / \hbar$, where $g$ is the $g$-factor. We have $g_e = -g_p = 2$ and so $\mu_e = -\frac{2 \mu_B}{\hbar} S_e$ and $\mu_p = \frac{2 \mu_B}{\hbar} S_p$. The Hamiltonian is

$$H = H_0 - A \mu_p \cdot \mu_e - (\mu_e + \mu_p) \cdot B = H_0 + \left(\frac{2 \mu_B}{\hbar}\right)^2 A S_p \cdot S_e + \left(\frac{2 \mu_B}{\hbar}\right) B (S_e^z - S_p^z).$$

(27.5.1)

Let $|1\rangle = |\uparrow\rangle$, $|2\rangle = |\downarrow\rangle$, $|3\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle + |\downarrow\rangle)$ and $|4\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle - |\downarrow\rangle)$, where the first spin is the electron and the second the positron. In the $|SmS\rangle$ basis, $|1\rangle = |11\rangle$, $|2\rangle = |1-1\rangle$, $|3\rangle = |10\rangle$, $|4\rangle = |00\rangle$.

$S_p \cdot S_e = \frac{1}{2} (S^2 - S_e^2 - S_p^2) = \frac{\hbar^2}{4} \left(\frac{2S^2}{\hbar^2} - 3\right)$, which is $\frac{\hbar^2}{4}$ for $S = 1$ and $-\frac{3\hbar^2}{4}$ for $S = 0$. Thus, $(\frac{2 \mu_B}{\hbar})^2 S_p \cdot S_e$ has the value $\mu_B^2$ on $S = 1$ states and $-3 \mu_B^2$ on $S = 0$ states.

Write the magnetic field term as $B \mu_B (\sigma_e^z - \sigma_p^z)$. It clear that this operator vanishes on $|1\rangle$ and $|2\rangle$. However, $\sigma_e^z |3\rangle = |4\rangle$ and $\sigma_p^z |3\rangle = -|4\rangle$, etc. Hence, in these bases, the Hamiltonian reads

$$H = \begin{pmatrix}
H_0 + A \mu_B^2 & 0 & 0 & 0 \\
0 & H_0 + A \mu_B^2 & 0 & 0 \\
0 & 0 & H_0 + A \mu_B^2 & 2B \mu_B \\
0 & 0 & 2B \mu_B & H_0 - 3A \mu_B^2
\end{pmatrix}.$$  

(27.5.2)

Forget $H_0$ since that does not affect diagonalization. We must diagonalize $\left(\frac{1}{2\beta} \begin{pmatrix} 2 & 2 \beta \\ 2 \beta & -3 \end{pmatrix}\right)$, where $\beta = B/A \mu_B$. The secular equation reads $(\lambda - 1)(\lambda + 3) - 4 \beta^2 = \lambda^2 + 2 \lambda - (3 + 4 \beta^2) = 0$ with solutions $\lambda_{\pm} = -1 \pm \sqrt{1 + 3 + 4 \beta^2} = -1 \pm 2 \sqrt{1 + \beta^2}$. Thus,

$$|1\rangle \text{ and } |2\rangle \text{ remain unchanged with energy } H_0 + A \mu_B^2,$$

$$|3\rangle \text{ transforms with energy } H_0 + A \mu_B^2 + 2A \mu_B^2 [\sqrt{1 + \beta^2} - 1],$$

$$|4\rangle \text{ transforms with energy } H_0 - 3A \mu_B^2 - 2A \mu_B^2 [\sqrt{1 + \beta^2} - 1].$$

(27.5.3)
Egs \leq \langle \psi | H | \psi \rangle \text{ for any normalized function } \psi.

WKB assumptions: in the classical region containing many wavelengths of oscillation, the potential is essentially constant so that the wavelength and amplitude of the wavefunction vary slowly; likewise, in the forbidden region, the amplitude and decay rate vary slowly.

In particular, if \( \psi(x) = A(x)e^{i\phi(x)} \), then \( A''/A << (\psi')^2 \) and \( k^2 = 2m(E - V(x))/\hbar^2 \).

Upward sloping turning point \( x = x_2 \):

\[
\psi(x) \approx \frac{D}{\sqrt{\hbar|k(x)|}} \left\{ \begin{array}{ll}
2 \sin \left[ \int_{x_2}^{x} k(x') \, dx' + \frac{\pi}{4} \right], & x < x_2, \\
\exp \left[ -\int_{x_2}^{x} |k(x')| \, dx' \right], & x > x_2.
\end{array} \right.
\]

Downward sloping turning point \( x = x_1 \):

\[
\psi(x) \approx \frac{D'}{\sqrt{\hbar|k(x)|}} \left\{ \begin{array}{ll}
\exp \left[ -\int_{x}^{x_1} |k(x')| \, dx' \right], & x < x_1, \\
2 \sin \left[ \int_{x_2}^{x_1} k(x') \, dx' + \frac{\pi}{4} \right], & x > x_1.
\end{array} \right.
\]

Connecting across turning point (ex: upward sloping): shift origin to the turning point and write \( V(x) \approx E + V'(0)x \), the S.E. becomes \( \psi'' = \alpha^2 x \psi \) where \( \alpha^2 = 2mV'(0)/\hbar^2 \). Then, defining \( z = \alpha x \) allows us to write this equation as \( \psi'' = z \psi \), which is Airy’s equation. The solution is \( \psi = a \text{Ai}(z) + b \text{Bi}(z) \) where \( \text{Ai}(z) \) and \( \text{Bi}(z) \) are the Airy functions.

Asymptotics of Airy functions:

\[
\text{Ai}(z) \sim \frac{1}{2\sqrt{\pi}|z|^{1/4}} \left\{ \begin{array}{ll}
\exp \left[ -\frac{2}{3} z^{3/2} \right], & z >> 0, \\
2 \sin \left[ \frac{2}{3} |z|^{3/2} + \frac{\pi}{4} \right], & z << 0.
\end{array} \right.
\]

\[
\text{Bi}(z) \sim \frac{1}{\sqrt{\pi}|z|^{1/4}} \left\{ \begin{array}{ll}
\exp \left[ \frac{2}{3} z^{3/2} \right], & z >> 0, \\
\cos \left[ \frac{2}{3} |z|^{3/2} + \frac{\pi}{4} \right], & z << 0.
\end{array} \right.
\]
28.1 Even Power Potential

[Spring 2003 Modern (Morning), Problem 2]

(a) Use the normalized Gaussian trial function \( \psi_t(x) = (\beta/\pi)^{1/4} e^{-\beta x^2/2} \) to obtain the best upper bound on the ground state energy for the \( n \)-dependent family of potentials \( V(x) = \lambda x^{2n} \), \( n = 1, 2, 3 \ldots \).

(b) Compare your bound with the known exact result for \( n = 1 \).

SOLUTION:

(a) Since the trial function is Gaussian, it vanishes sufficiently fast at \( \infty \) to justify integration by parts and neglecting surface terms. That is,
\[
E_t = \int_{-\infty}^{\infty} dx \left( \frac{\hbar^2}{2m} \frac{d^2 \psi_t}{dx^2} + V(x) |\psi_t(x)|^2 \right) \\
= \sqrt{\frac{\beta}{\pi}} \int_{-\infty}^{\infty} dx \ e^{-\beta x^2} \left[ \frac{\hbar^2 \beta^2}{2m} x^2 + \lambda x^{2n} \right] \\
= \frac{\hbar^2 \beta}{4m} + \frac{\lambda (2n)!}{(4\beta^n)n!},
\]
where we used the result \( \int_{-\infty}^{\infty} dx \ x^{2n} e^{-x^2} = \frac{(2n)!\sqrt{\pi}}{2^n n!} \), which you can easily prove by induction:
\[
\int_{0}^{\infty} dx \ x^{2n} e^{-x^2} = -\frac{1}{2} e^{-x^2} x^{2n-1} \bigg|_{0}^{\infty} + \frac{2n-1}{2} \int_{0}^{\infty} dx \ x^{2(n-1)} e^{-x^2}. \tag{28.1.2}
\]

Let \( J(n) \equiv \int_{0}^{\infty} dx \ x^{2n} e^{-x^2} \), then, this says
\[
J(n) = \frac{2n-1}{2} J(n-1) = \frac{1}{2^n n!} (2n)! \left( \frac{2(n-1)!}{n!} \right) J(n-1). \tag{28.1.3}
\]

The purpose of this latter form is that, defining \( \alpha(n) = \frac{(2n)!}{2^n n!} \), we have
\[
J(n) = \frac{\alpha(n)}{\alpha(n-1)} J(n-1). \tag{28.1.4}
\]

Generically, one wants to put a recursion formula in such a form so that we get
\[
J(n) = \frac{\alpha(n)}{\alpha(n-1)} \frac{\alpha(n-1)}{\alpha(n-2)} \cdots \frac{\alpha(4)}{\alpha(3)} \frac{\alpha(3)}{\alpha(2)} \frac{\alpha(2)}{\alpha(1)} J(0) = \frac{\alpha(n)}{\alpha(0)} J(0). \tag{28.1.5}
\]

Knowing \( J(0) = \sqrt{\pi}/2 \) and calculating \( \alpha(0) = 1 \), we find
\[
J(n) = \frac{(2n)! \sqrt{\pi}}{2^{2n} n!} = \frac{(2n)! \sqrt{\pi}}{2^{2n+1} n!}. \tag{28.1.6}
\]

The integral we want is just twice this since it goes from \(-\infty\) to \(+\infty\).
Okay, now we must minimize the result (28.1.1) with respect to the only free parameter, $\beta$. Setting the derivative $\partial E_t/\partial \beta$ to zero yields a minimum at

$$
\beta_0 = \left( \frac{\lambda m (2n)!}{\hbar^2 2^{(n-1)} (n-1)!} \right)^{\frac{1}{2n+1}},
$$  

(28.1.7)

at which point, the energy bound is

$$
E_0 \leq E_t(\beta_0) = \frac{\hbar^2}{4m} \left( 1 + \frac{1}{n} \right) \left( \frac{\lambda m (2n)!}{\hbar^2 2^{(n-1)} (n-1)!} \right)^{\frac{1}{2n+1}}.
$$  

(28.1.8)

(b) When $n = 1$, we get $E_0 \leq \frac{\hbar}{2} \sqrt{2\lambda/m} = \frac{\hbar}{2} \omega$ since, for a harmonic oscillator, $V(x) = \frac{1}{2} m \omega^2 x^2$ and thus, since $V(x) = \lambda x^2$, we have $2\lambda/m = \omega^2$. Therefore, the variational result is exact in this case.
28.2 Half-Airy Potential

Spring 2004 Modern (Afternoon), Problem 5

(a) Use the variational method and the trial function $x^p e^{-ax}$ (for $x > 0$ and 0 for $x < 0$), where $a > 0$ and $p$ is a positive integer, to estimate the ground state energy of a particle in the one-dimensional potential

$$V(x) = \begin{cases} \infty, & x < 0, \\ Cx, & x \geq 0, \end{cases}$$

where $C$ is a constant. First choose the optimum value of $p$. Explain why the above trial function is appropriate.

(b) Under what condition can the variational theorem be applied to the first excited state? The trial function

$$\phi_0(x) = \begin{cases} a^2 - x^2, & |x| < a, \\ 0, & |x| \geq a, \end{cases}$$

gives a good estimate of the ground state energy of a one-dimensional harmonic oscillator. Explain why the function $\phi_1 = x\phi_0$ is a suitable trial function for the first excited state of the harmonic oscillator.

SOLUTION:

(a) Certainly, these trial functions satisfy the condition that $\psi(x \leq 0) = \psi(x \to \infty) = 0$, which is required since due to the infinite-ness of the potential. But, ultimately, the reason we like these functions is that they’re relatively easy to integrate.

Since the trial functions are not normalized, let’s first normalize them. For this, it is convenient to first accumulate some important integrals. Define $J(n) \equiv \int_0^\infty dx x^n e^{-x}$ and integrate by parts

$$J(n) = -x^n e^{-x}\big|_0^\infty + nJ(n-1) = \frac{n!}{(n-1)!}J(n-1). \quad (28.2.1)$$

Therefore, $J(n) = n!$, since $J(0) = 1$. Thus, we have

$$I(n) \equiv \int_0^\infty dx x^n e^{-2ax} = \frac{n!}{(2a)^{n+1}}. \quad (28.2.2)$$

Alternatively, you could have taken $n$ derivatives of $I(0)$ with respect to $a$.

It follows that the properly normalized trial functions are

$$\psi(x) = N x^p e^{-ax}, \quad \text{where} \quad N^2 = \frac{(2a)^{2p+1}}{(2p)!}. \quad (28.2.3)$$

After some algebra, one finds

$$\langle \psi | H | \psi \rangle = N^2 \int_0^\infty dx e^{-2ax} \left[ -\frac{\hbar^2}{2m} (p(p-1)x^{2p-2} - 2apx^{2p-1} + a^2 x^{2p}) + Cx^{2p+1} \right]. \quad (28.2.4)$$

Performing each of the integrals using $(28.2.2)$ and $N$ yields

$$E = \langle \psi | H | \psi \rangle = \frac{1}{2p - 1} \frac{\hbar^2 a^2}{2m} + \frac{2p + 1}{2a} C. \quad (28.2.5)$$
28.2. HALF-AIRY POTENTIAL

Firstly, the professor calculates this energy incorrectly (his or her units are not even correct!) Secondly, the professor makes the seemingly innocent “guess” that the minimal energy bound is achieved by \( p = 1 \) since the probability distribution is focused at lower values of \( x \) for lower values of \( p \) and the that is where the potential energy is minimal as well. While it’s well-motivated, it’s also plain WRONG!

Setting the partial derivatives of \( E \) with respect to \( p \) and \( a \) to zero yields the two equations

\[
\begin{align*}
    a^3 &= \frac{mc}{\hbar^2} (2p - 1)^2, \\
    a^3 &= \frac{mc}{2\hbar^2} (2p - 1)(2p + 1).
\end{align*}
\]

(28.2.6)

It’s quite easy to find the solution:

\[
\begin{align*}
    p_0 &= \frac{3}{2}, \\
    a^3_0 &= \frac{4mC}{\hbar^2}.
\end{align*}
\]

(28.2.7)

Of course, \( p_0 \) is not an integer here, which is a restriction that I find unnecessary since non-integer values for \( p \) still give well-defined functions! You see, if we were to believe the “innocent guess” and allowed \( p \) to be non-integral, then we would have thought a much smaller value of \( p \) would be preferable, but, in fact, the real minimum is bigger than \( p = 1 \). Since there is only one positive extremum, it had better be a minimum and \( E \) can only monotonically increase away from this minimum. If we need integral \( p \), then it follows that the minimum must occur either at \( p = 1 \) or \( p = 2 \). For the two cases, we get

\[
\begin{align*}
    E &= \frac{\hbar^2 a^2}{2m} + \frac{3C}{2a}, \\
    E &= \frac{\hbar^2 a^2}{6m} + \frac{5C}{2a}.
\end{align*}
\]

(28.2.8)

The corresponding minimal \( a \) are

\[
\begin{align*}
    a^3_0 &= \frac{3mC}{2\hbar^2}, \\
    a^3_0 &= \frac{15mC}{2\hbar^2}.
\end{align*}
\]

(28.2.9)

Plugging these into (28.2.8) gives

\[
\begin{align*}
    E &= (3/2)^{5/3}(\hbar^2 C^2/m)^{1/3}, \\
    E &= (5^{2/3}/3)(3/2)^{5/3}(\hbar^2 C^2/m)^{1/3}.
\end{align*}
\]

(28.2.10)

Since \( 5^{2/3}/3 < 1 \), the second energy, which came from \( p = 2 \) is actually lower than the first energy, which came from \( p = 1 \). Thus, the bound for the ground state energy is

\[
E_0 \leq (5^{2/3}/3)(3/2)^{5/3}(\hbar^2 C^2/m)^{1/3} \approx 1.92(\hbar^2 C^2/m)^{1/3}.
\]

(28.2.11)

(b) One can use the variational principle for the first excited state if we use a trial function that is orthogonal to the ground state. We can do this for the harmonic oscillator, for example, if we use an odd function as a trial function for the first excited state since the ground state is even and so the trial function and the ground state are automatically orthogonal. In the present case, \( \phi_0 \) is even and \( \phi_1 = x\phi_0 \) is odd and is thus suitable as a trial function for the first excited state.
28.3 Vibrating String

[Zwiebach 4.5] Consider a string stretched from $x = 0$ to $x = a$, with a constant tension $T_0$ and a position-dependent mass density $\mu(x)$. The string is fixed to $y = 0$ at the endpoints and can vibrate in the $y$ direction.

(a) Set up a variational procedure giving an upper bound on the lowest oscillation frequency, $\omega_0$. [Hint: Usually, the inner product is $\langle \psi_i | \psi_j \rangle \equiv \int \psi_i^*(x) \psi_j(x) \, dx$. But, you could stick a sufficiently well-behaved, non-negative function in their too. Consider $\langle \psi_i | \psi_j \rangle \equiv \int \mu(x) \psi_i^*(x) \psi_j(x) \, dx$.]

(b) For $\mu(x) = \mu_0 x a$, use your variational principle to find a simple bound on the lowest oscillation frequency. Compare with the real answer $\omega_0^2 \approx (18.956) \frac{T_0}{\mu_0 a^2}$ obtained by computer.

SOLUTION:

(a) Consider the diagram below:

![Vibrating string diagram]

Figure 28.1: Vibrating string.

The vertical force is $dF_v = T_0 \left( \frac{\partial y}{\partial x} \right)_{x+dx} - \frac{\partial y}{\partial x} \right)_{x} \approx T_0 \frac{\partial^2 y}{\partial x^2} \, dx$. The mass in this bit is $dm = \mu(x) \, dx$. Thus, Newton’s third law reads $T_0 \frac{\partial^2 y}{\partial x^2} - \frac{\partial^2 y}{\partial t^2} = 0$. Set $y(x,t) = y(x) \sin(\omega t + \phi)$ in terms of which any arbitrary solution may be expanded. Then,

$$y'' + \mu(x) \frac{\omega^2}{T_0} y(x) = 0.$$  \hfill (28.3.1)

Now, suppose this is quantized. Replace $y$ with $\psi_i$ with corresponding frequency $\omega_i$. Since the $y$'s are, we may take the $\psi_i$’s to be real. Define $\Omega^2 = -\frac{T_0}{\mu(x)} \frac{d^2}{dx^2}$, so that $\Omega^2 \psi_i = \omega_i^2 \psi_i$.

The first thing we will show is that these states are orthogonal with respect to the inner product proposed in the problem. This is a consequence of the Sturm-Liouville theorem, but it’s easy enough to derive.

$$\langle \psi_i | \tilde{\pi} | \psi_j \rangle = \int_0^a \mu(x) \psi_i(x) \psi_j(x) \, dx = -\frac{T_0}{\omega_j^2} \int_0^a \psi_i(x) \frac{d^2 \psi_j}{dx^2} \, dx.$$  \hfill (28.3.2)

Integrate by parts twice. The boundary terms vanish since $\psi_i(0) = \psi_i(a) = 0$. Thus, we get

$$\langle \psi_i | \psi_j \rangle = \left( \frac{\omega_j}{\omega_i} \right)^2 \langle \psi_i | \psi_j \rangle.$$  \hfill (28.3.3)

Thus, $\langle \psi_i | \psi_j \rangle = \langle \psi_i | \psi_i \rangle \delta_{ij}$, proving orthogonality.

Expand an arbitrary function $\psi = \sum c_n \psi_n$. Then,

$$\langle \psi | \Omega^2 | \psi \rangle = \sum_{mn} c_m c_n \langle \psi_m | \Omega^2 | \psi_n \rangle = \sum_n \omega_n^2 |c_n|^2 \langle \psi_n | \psi_n \rangle \geq \omega_0^2 \langle \psi | \psi \rangle.$$  \hfill (28.3.4)
The usual variational result follows:

\[
\omega_0^2 \leq \frac{\langle \psi | \Omega^2 | \psi \rangle}{\langle \psi | \psi \rangle} \tag{28.3.5}
\]

(b) Take \( \psi = A \sin \left( \frac{\pi x}{a} \right) \), which would be the ground state if the mass density were constant. We have \( \Omega^2 \psi = \frac{T_0}{\mu(x)} \left( \frac{\pi}{a} \right)^2 A \sin \left( \frac{\pi x}{a} \right) \). Thus,

\[
\langle \psi | \Omega^2 | \psi \rangle = \int_0^a \mu(x) A \sin \left( \frac{\pi x}{a} \right) \frac{T_0}{\mu(x)} \left( \frac{\pi}{a} \right)^2 A \sin \left( \frac{\pi x}{a} \right) dx
\]

\[
= \frac{\pi^2 A^2 T_0}{a^2} \int_0^a \sin^2 \left( \frac{\pi x}{a} \right) dx
\]

\[
= \frac{\pi A^2 T_0}{a} \int_0^\pi \sin^2 \xi d\xi
\]

\[
= \frac{\pi^2 A^2 T_0}{2a}. \tag{28.3.6}
\]

Similarly, we find

\[
\langle \psi | \psi \rangle = \frac{\mu_0 A^2}{a} \int_0^a x \sin^2 \left( \frac{\pi x}{a} \right) dx
\]

\[
= \frac{\mu_0 A^2 a}{\pi^2} \int_0^\pi \xi \sin^2 \xi d\xi
\]

\[
= \frac{\mu_0 A^2 a}{4}. \tag{28.3.7}
\]

Therefore, we have the bound

\[
\omega_0^2 \leq \frac{\langle \psi | \Omega^2 | \psi \rangle}{\langle \psi | \psi \rangle} = \frac{(2\pi^2) T_0}{\mu_0 a^2} = \frac{(19.739) T_0}{\mu_0 a^2} \tag{28.3.8}
\]

Note: Using \( \psi = -Ax(x-a) \) gives 20 instead of 19.739.
28.4 WKB Derivation

[Griffiths 8.2] An illuminating alternative derivation of the WKB formula, \( \psi(x) \sim \frac{C}{\sqrt{p(x)}} \exp\left[ \pm \frac{i}{\hbar} \int p(x) \, dx \right] \), is based on an expansion in powers of \( \hbar \). Motivated by the free particle wave function, \( \psi = A e^{\pm ikx} \), we write \( \psi(x) = e^{iF(x)/\hbar} \), where \( F(x) \) is some complex function. [Note that there is no loss of generality here - any nonzero function can be written in this way.]

(a) Put this into Schrödinger’s equation and show that \( i\hbar f'' - (f')^2 + p^2 = 0 \).

(b) Write \( f(x) \) as a power series in \( \hbar \): \( f(x) = \sum_{n=0}^{\infty} \hbar^n f_n(x) \), and, collecting like powers of \( \hbar \), show that

\[
(f_0')^2 = p^2, \quad if_0'' = 2f_0'f_1', \quad if_1'' = 2f_0'f_1' + (f_1')^2, \quad \text{etc.}
\]

(c) Solve for \( f_0(x) \) and \( f_1(x) \), and show that - to first order in \( \hbar \) - you recover the WKB result.

Note: The logarithm of a negative number is defined by \( \ln(z) = \ln(|z|) + i\pi n \) where \( n \) is an odd integer. If this formula is new to you, try exponentiating both sides, and you’ll see where it comes from.

**SOLUTION:**

(a) Schrödinger’s equation can be put in the form \( \left[ \frac{d^2}{dx^2} + \frac{p^2}{\hbar^2} \right] \psi = 0 \), where \( p(x)^2 = 2m(E - V(x)) \).

Plugging in \( \psi \) gives \( \left[ \frac{d}{dx} f'' - \frac{1}{\hbar^2} (f')^2 + \frac{p^2}{\hbar^2} \right] e^{iF/\hbar} = 0 \), which implies the desired result.

(b) This is just plug and chug:

\[
i \sum_{n=0}^{\infty} \hbar^{n+1} h^n f_n'' - \sum_{m,n=0}^{\infty} \hbar^{m+n} f_m f_n' + p^2 = 0. \tag{28.4.1}
\]

The \( h^0 \) terms read \( -(f_0')^2 + p^2 = 0 \). The \( h^1 \) terms read \( if_0'' - f_0'f_1' - f_1'f_0' = 0 \), or \( if_0'' = 2f_0'f_1' \).

The \( h^2 \) terms read \( if_1'' - f_0''f_2' - f_1''f_0' - f_2'f_0'' = 0 \) or \( if_1'' = 2f_0''f_2' + (f_2')^2 \).

(c) From the \( h^0 \) equation, we get \( f_0' = \pm p \) or

\[
f_0 = \pm \int p(x) \, dx + \text{const.} \tag{28.4.2}
\]

From the \( h^1 \) equation, \( f_1' = if_1''/2f_0' = \frac{i}{2}(\pm p'/ \pm p) = \frac{i}{2} \frac{d}{dx} \ln p \). Hence, \( f_1 = \frac{i}{2} \ln p + \text{const.} \). Therefore,

\[
\psi \approx \exp \left[ \frac{i}{\hbar} (f_0 + \hbar f_1) \right] = \frac{C}{\sqrt{p}} \exp \left( \pm \frac{i}{\hbar} \int p(x) \, dx \right). \tag{28.4.3}
\]
28.5  Barrier with Sloping Walls

[Griffiths 8.10] Use appropriate connection formulas to analyze the problem of scattering from a barrier with sloping walls with upward sloping turning point at \( x_1 \), downward turning point \( x_2 \) and no other classical turning points.

**SOLUTION:**

Consider first the upward sloping turning point at \( x_1 \) and set the origin to be at \( x_1 \). Assuming WKB may be applied, we write the wavefunction on either side as

\[
\psi(x) \approx \frac{1}{\sqrt{h|k(x)|}} \begin{cases} 
A \exp \left[ \int_{x}^{x_1} k(x') \, dx' \right] + B \exp \left[ -i \int_{x}^{x_1} k(x') \, dx' \right], & x < 0, \\
C \exp \left[ \int_{x}^{0} |k(x')| \, dx' \right] + D \exp \left[ -i \int_{x}^{0} |k(x')| \, dx' \right], & x > 0.
\end{cases}
\]  

(28.5.1)

If we expand \( V(x) \approx E + V'(0)x \), then \( k(x) \approx \frac{1}{\hbar} \sqrt{-2mV'(0)x} = \alpha^{3/2} \sqrt{-x} \), where \( \alpha^{3} = 2mV'(0)/\hbar^2 \). Thus, \( \int_{0}^{x} |k(x')| \, dx' \approx \alpha^{3/2} \int_{0}^{x} \sqrt{x'} \, dx' = \frac{2}{3} \alpha^{3/2} \), where \( z = \alpha x \). In the region \( x > 0 \) (or \( z > 0 \)), we have

\[
\psi(z) \approx (\hbar \alpha \sqrt{z})^{-1/2} \left[ C \exp \left( \frac{2}{3} \alpha^{3/2} \right) + D \exp \left( -\frac{2}{3} \alpha^{3/2} \right) \right].
\]

(28.5.2)

Meanwhile, if we write the wavefunction around the turning point in terms of Airy functions, \( \psi(z) = a \text{Ai}(z) + b \text{Bi}(z) \), then the \( z \gg 0 \) limit reads

\[
\psi(z) \approx (\pi \sqrt{z})^{-1/2} \left[ \frac{a}{2} \exp \left( -\frac{2}{3} \alpha^{3/2} \right) + b \exp \left( \frac{2}{3} \alpha^{3/2} \right) \right].
\]

(28.5.3)

Matching coefficients yields

\[
a = 2D \sqrt{\frac{\pi}{\alpha \hbar}}, \quad b = C \sqrt{\frac{\pi}{\alpha \hbar}}.
\]

(28.5.4)

On the other hand, in the region \( x < 0 \), we have

\[
\psi(z) \approx (\hbar \alpha \sqrt{|z|})^{-1/2} \left[ A \exp \left( i \frac{2}{3} |z|^{3/2} \right) + B \exp \left( -i \frac{2}{3} |z|^{3/2} \right) \right],
\]

(28.5.5)

and the Airy function limit is

\[
\psi(z) \approx (\pi \sqrt{|z|})^{-1/2} \left[ a \sin \left( \frac{2}{3} |z|^{3/2} \pi \right) + b \cos \left( \frac{2}{3} |z|^{3/2} \pi \right) \right]
\]

\[
= \frac{1}{2} (\pi \sqrt{|z|})^{-1/2} \left[ (-ia + b)e^{i\pi/4} \exp \left( i \frac{2}{3} |z|^{3/2} \right) + (ia + b)e^{-i\pi/4} \exp \left( -i \frac{2}{3} |z|^{3/2} \right) \right].
\]

(28.5.6)

Matching coefficients yields

\[
A = \sqrt{\frac{\hbar \alpha}{\pi}} \left( \frac{-ia + b}{2} \right) e^{i\pi/4}, \quad B = \sqrt{\frac{\hbar \alpha}{\pi}} \left( \frac{ia + b}{2} \right) e^{-i\pi/4}.
\]

(28.5.7)

Combining (28.5.4) and (28.5.7) yields

\[
A = \left( \frac{1}{2} C - iD \right) e^{i\pi/4}, \quad B = \left( \frac{1}{2} C + iD \right) e^{-i\pi/4}.
\]

(28.5.8)

We must repeat this exercise for the downward sloping turning point at \( x_2 \). Set the origin at \( x_2 \). Let us rewrite the wavefunction in the middle as

\[
\psi(x) \approx \frac{1}{\sqrt{h|k(x)|}} \left[ C' \exp \left( \int_{x}^{0} |k(x')| \, dx' \right) + D' \exp \left( -\int_{x}^{0} |k(x')| \, dx' \right) \right],
\]

(28.5.9)
where
\[ \gamma \equiv \int_{x_1}^{x_2} |k(x)| \, dx, \quad C' = De^{-\gamma}, \quad D' = Ce^\gamma. \quad (28.5.10) \]
As before, we have
\[ \psi(z) \approx [\hbar \alpha \sqrt{|z|}]^{-1/2} \left[ C' \exp\left(\frac{2}{3} |z|^{3/2}\right) + D' \exp\left(-\frac{2}{3} |z|^{3/2}\right) \right], \quad (28.5.11) \]
whereas the asymptotic Airy function reads
\[ \psi(z) \approx [\pi \sqrt{|z|}]^{-1/2} \left[ \frac{a}{2} \exp\left(-\frac{2}{3} |z|^{3/2}\right) + b \exp\left(\frac{2}{3} |z|^{3/2}\right) \right], \quad (28.5.12) \]
and thus, matching coefficients yields
\[ a = 2D' \sqrt{\frac{\pi}{\alpha \hbar}}, \quad b = C' \sqrt{\frac{\pi}{\alpha \hbar}}. \quad (28.5.13) \]
Write the wavefunction for \( x > 0 \) as
\[ \psi(z) \approx \frac{1}{\sqrt{\hbar |k(x)|}} \left[ F \exp\left( i \int_0^x k(x') \, dx' \right) \right] \]
\[ \approx [\hbar \alpha \sqrt{|z|}]^{-1/2} F \exp\left[i \frac{2}{3} z^{3/2}\right]. \quad (28.5.14) \]
The Airy function version reads
\[ \psi(z) \approx (\pi \sqrt{|z|})^{-1/2} \left[ a \sin\left(\frac{2}{3} z^{3/2} + \frac{\pi}{3}\right) + b \cos\left(\frac{2}{3} z^{3/2} + \frac{\pi}{3}\right) \right] \]
\[ = \frac{1}{2} (\pi \sqrt{|z|})^{-1/2} \left[ (-ia + b)e^{i\pi/4} \exp\left(i \frac{2}{3} z^{3/2}\right) + (ia + b)e^{-i\pi/4} \exp\left(-i \frac{2}{3} z^{3/2}\right) \right]. \quad (28.5.15) \]
Matching coefficients implies that \( ia + b = 0 \) and
\[ a = i \sqrt{\frac{\pi}{\alpha \hbar}} e^{-i\pi/4} F. \quad (28.5.16) \]
Combining all of the coefficient relations together gives
\[ A = i \left( \frac{1}{4} e^{-\gamma} - e^{\gamma} \right) F, \quad (28.5.17) \]
and thus, the transmission coefficient is
\[ T = \frac{|F|^2}{|A|^2} = \frac{e^{-2\gamma}}{\left[ 1 - (e^{-\gamma}/2)^2 \right]^2}. \quad (28.5.18) \]
Chapter 29

Time-Dependent Perturbation Theory

• Expand $|\psi(t)\rangle$ in the eigenbasis for the unperturbed Hamiltonian: $|\psi(t)\rangle = \sum_n c_n(t)e^{-iE_n t/\hbar}|n\rangle$. Plug into Schrödinger’s equation and take the inner product with $|m\rangle$ to get $\dot{c}_m(t) = -i\sum_n H'_{mn} e^{i\omega_{mn} t} c_n$, where $H'_{mn} = \langle m|H'|n\rangle$ and $\omega_{mn} = (E_m - E_n)/\hbar$.

• Order $k$ equation: Write $H = H_0 + \lambda H'$ and $c_m = \sum_{n=0}^{\infty} \lambda^n c_m^{(n)}$, then $\dot{c}_m^{(k)} = -i\sum_n H'_{mn} e^{i\omega_{mn} t} c_n^{(k-1)}$.

• Dipole radiation perturbation: $H' = qEz \cos(\omega t)$ and we require $\langle m|z|n\rangle$.

• Stimulated emission by incoherent, unpolarized, non-monochromatic radiation: $R_{b\rightarrow a} = \frac{2\pi}{\hbar}\rho(\omega_0)|p|^2\rho(\omega_0)$, where $p = qr$ and $\omega_0 = (E_b - E_a)/\hbar$ and $\rho(\omega)d\omega$ is the energy density within the frequency range $d\omega$.

• Selection rules for dipole transitions: $\Delta m = \pm 1$ or $0$ and $\Delta \ell = \pm 1$ are the only allowed transitions.

• Fermi’s golden rule: transition rate from one initial to one particular final state is $w_1 = \frac{2\pi}{\hbar}|\langle f|V|i\rangle|^2\delta(E_f - E_i)$. The transition rate to a continuum of final states with density of states $\rho_f(E_f)$ is $w = \int dE_f \rho_f(E_f)w_1 = \frac{2\pi}{\hbar}|\langle f|V|i\rangle|^2\rho_f(E_i)$.

• Transition rate and cross section are related via $dw = nv d\sigma$, where $n$ is the volume density of incoming particles (usually taken to be $1/V$ for single scattering) and $v$ is the incoming speed.
29.1 Perturbed One-Dimensional Box

[Fall 2008 Modern (Morning), Problem 3] A particle of mass \(m\) is confined in an infinite square well potential \(V(x) = 0\) for \(0 \leq x \leq b\) and \(V(x) = \infty\) otherwise.

(a) Find the eigenvalues and eigenfunctions of this particle.

(b) Add \(V_1(x) = \epsilon \sin \frac{\pi}{b} x\) to \(V\). Find the energy shifts for all the excited states to first order in \(V_1\).

(c) Replace \(V_1\) with \(V_2(x, t) = \lambda (x - \frac{b}{2}) \sin \omega t\). If the particle is in the ground state at \(t = 0\), what is the probability that the particle is in the first excited state at some later time \(t\)? What about the second excited state?

SOLUTION:

(a) The solution to a free particle is a sum of cos and sin. For \(\psi\) to vanish at \(x = 0\) and \(x = b\), as well as be properly normalized, \(\psi_n(x) = \sqrt{\frac{2}{b}} \sin \left(\frac{n\pi x}{b}\right)\) with corresponding energies

\[
E_n = \frac{\pi^2 \hbar^2 n^2}{2mb^2}, \quad \text{for } n = 1, 2, \ldots
\]

(b) This is first order non-degenerate time-independent perturbation theory:

\[
\Delta E_n = \langle n|V_1|n \rangle = \frac{2\epsilon}{b} \int_0^b \sin \left(\frac{\pi x}{b}\right) \sin^2 \left(\frac{n\pi x}{b}\right) dx
\]

\[
= \frac{2\epsilon}{\pi} \int_0^\pi \sin(\xi) \sin^2(n\xi) d\xi
\]

\[
= \frac{\epsilon}{\pi} \int_0^\pi \sin \xi d\xi - \frac{\epsilon}{\pi} \int_0^\pi \sin(2n\xi) d\xi
\]

\[
= \frac{2\epsilon}{\pi} - \frac{\epsilon}{\pi} \int_0^\pi \sin(\xi) \cos(2n\xi) d\xi.
\]  (29.1.1)

Let us compute this integral by integration by parts:

\[
\int_0^\pi \sin(\xi) \cos(2n\xi) d\xi = -\cos(\xi) \cos(2n\xi)|_0^\pi - \int_0^\pi (-\cos(\xi)(-2n) \sin(2n\xi)) d\xi
\]

\[
= 2 - 2n \int_0^\pi \cos(\xi) \sin(2n\xi) d\xi
\]

\[
= 2 - 2n \left[ \sin(\xi) \sin(2n\xi) \right|_0^\pi - \int_0^\pi \sin(\xi)(2n) \cos(2n\xi) d\xi \right]
\]

\[
= 2 + 4n^2 \int_0^\pi \sin(\xi) \cos(2n\xi) d\xi.
\]  (29.1.2)

It follows that

\[
\Delta E_n = \frac{2\epsilon}{\pi} \left( 1 + \frac{1}{4n^2} - 1 \right).
\]  (29.1.3)
29.1. **PERTURBED ONE-DIMENSIONAL BOX**

(c) Using the same procedure as above, for \( n \neq 1 \), we find the integrals

\[
\int_0^\pi \sin(\xi) \sin(n\xi) \, d\xi = \frac{1}{n^2 - 1} [1 - (-1)^{n+1}],
\]

\[
\int_0^\pi \cos(\xi) \cos(n\xi) \, d\xi = \frac{1}{n^2 - 1},
\]

\[
\int_0^\pi \sin(\xi) \sin(n\xi) \, d\xi = 0,
\]

\[
\int_0^\pi \xi \sin(\xi) \sin(n\xi) \, d\xi = [(-1)^{n+1} - 1] \frac{2n}{(n^2 - 1)^2}.
\]

Then, we can compute the inner product:

\[
\langle n | V_2 | 1 \rangle = \frac{2\lambda}{b} \sin \omega t \int_0^b \left( x - \frac{b}{2} \right) \sin \left( \frac{\pi x}{b} \right) \sin \left( \frac{\pi x}{b} \right) \, dx
\]

\[
= \frac{b\lambda}{\pi} \sin \omega t \left[ \frac{2}{\pi} \int_0^\pi \xi \sin(\xi) \sin(n\xi) \, d\xi - \int_0^\pi \sin(\xi) \sin(n\xi) \, d\xi \right]
\]

\[
= [(-1)^{n+1} - 1] \frac{n}{(n^2 - 1)^2} \frac{4b\lambda}{\pi^2} \sin \omega t.
\] (29.1.4)

If \( n \) is odd, then \( P_{1 \rightarrow n} = 0 \). Thus, \( P_{1 \rightarrow 3} = 0 \).

Expand \( |\psi(t)\rangle = \sum_m c_m(t) e^{-i\omega_m t} |m\rangle \). Plug this back into Schrödinger’s equation and take the inner product with \( \langle n | \). The resulting equation reads

\[
\dot{c}_n = -\frac{i}{\hbar} \sum_m \langle n | V_2 | m \rangle e^{i\omega_m t} c_m.
\] (29.1.5)

The zeroth order result is \( c_m^{(0)} = \delta_{m1} \). We plug this in the right hand side and solve for the first order corrected coefficient on the left:

\[
\dot{c}_n^{(1)} = -\frac{i}{\hbar} \langle n | V_2 | 1 \rangle e^{i\omega_1 t}.
\] (29.1.6)

Plugging in the inner product and time integrating yields

\[
c_n(t) = -\frac{i}{\hbar} [(-1)^{n+1} - 1] \frac{n}{(n^2 - 1)^2} \frac{4b\lambda}{\pi^2} \int_0^t e^{i\omega_{nm} t'} \sin \omega t' \, dt'
\]

\[
= \cdots \frac{1}{2} e^{i(\omega_{n1} - \omega) t} \left[ \frac{e^{i(\omega_{n1} + \omega) t}}{\omega_{n1} + \omega} - \frac{e^{i(\omega_{n1} - \omega) t}}{\omega_{n1} - \omega} \right].
\] (29.1.7)

Plug in \( \omega_{21} = \omega_2 - \omega_1 = 3\omega_1 = \frac{3\pi^2 \hbar}{2m b^2} \) for \( n = 2 \).
29.2 Two-Level Atom

[Fall 2002 Modern (Morning), Problem 6] Consider charged particles directed towards an atom. Model the atom as a two-level system with ground state |1⟩ and excited state |2⟩. The Hamiltonian matrix of the atom is

\[ H = \begin{pmatrix} 0 & -eE_{12} \\ -eE_{12} & \epsilon \end{pmatrix}, \]

where \( \epsilon \) is the energy of the excited state relative to the ground state, \( E \) is the electric field due to the charged particles, and \( r_{12} \equiv \langle 1|r|2 \rangle \). Here \( r \) is the component of the electron position vector in the direction of the \( E \) field.

The state of the two level system evolves with time according to

\[ |\psi(t)\rangle = c_1(t)|1\rangle + c_2(t)e^{-i\omega_0 t}|2\rangle, \]

where \( \omega \equiv \epsilon/\hbar \), and the coefficients satisfy the coupled rate equations

\[ \dot{c}_1 = -\frac{i}{\hbar}qE_{12}e^{-i\omega_0 t}c_2, \quad \dot{c}_2 = -\frac{i}{\hbar}qE_{12}e^{i\omega_0 t}c_1. \]

Assume that the atom experiences a delta-function electric field of the form \( E(t) = \alpha \delta(t - t_0) \) due to the passage of a charged particle.

(a) What is the dimensionless parameter that, if small, justifies the use of perturbation theory to characterize the response of the atom to a particle?

(b) Assuming that the parameter specified in (a) is indeed small, and that the atom is initially in its ground state, what is the probability that it will be in the excited state after passage of a particle? [To lowest order in \( \alpha \).

(c) What is the expectation value of the dipole moment of the atom, or \( \langle -er \rangle \), after passage of a particle, to lowest order in \( \alpha \)?

(d) Imagine that the atom is subject to \( N \) particles which arrive at equal intervals \( \tau \), where \( \omega_0 \tau << 1 \). Find the probability that the atom will be in the excited state after passage of the \( N \) particles [again to lowest order in \( \alpha \).]

SOLUTION:

(a) From the rate equation,

\[ c_2(t) = -\frac{iq r_{12} \alpha}{\hbar} \int_{-\infty}^{t} dt' \delta(t' - t_1)e^{i\omega_0 t'}c_1(t') + \text{const.} = -\frac{iq r_{12} \alpha}{\hbar} e^{i\omega_0 t_1}c_1(t_1) + \text{const.} \quad (29.2.1) \]

Therefore, the parameter that we need to keep small is \( |qr_{12}\alpha/\hbar| \).

(b) The intial condition \( c_1(0) = 1 \) and \( c_2(0) = 0 \) fixes the constant in part (a) to 0. Hence, \( c_2(t > t_1) = -(iqr_{12}\alpha/\hbar)e^{i\omega_0 t_1} \) and

\[ P_2 = |c_2|^2 = (qr_{12}\alpha/\hbar)^2. \quad (29.2.2) \]

(c) There are only off-diagonal elements: \( \langle r \rangle = 2 \text{Re}[r_{12}c_1^*c_2e^{-i\omega_0 t}] \). The first order correction only affects \( c_2 \) and \( c_1 \approx 1 + O(\alpha^2) \). Hence,

\[ \langle r \rangle = 2r_{12} \text{Re}\left(-\frac{iqr_{12}\alpha}{\hbar}e^{i\omega_0 t_1}e^{-i\omega_0 t} \right) = -\frac{2q r_{12}^2}{\hbar} \sin[\omega_0(t - t_1)]. \quad (29.2.3) \]
(d) We just perturb $c_2$ successively assuming that the overall perturbation is still small:

$$
c_2(N\tau) = -\frac{i\alpha q r_{12}}{\hbar} \sum_{n=0}^{N-1} e^{i\omega_0 n\tau} = -\frac{i\alpha q r_{12}}{\hbar} \frac{1 - e^{i\omega_0 N\tau}}{1 - e^{i\omega_0 \tau}}
= -\frac{i\alpha q r_{12}}{\hbar} e^{i(N-1)\omega_0 \tau/2} \frac{\sin(\omega_0 N\tau/2)}{\sin(\omega_0 \tau/2)}. \quad (29.2.4)
$$

Therefore, the probability that the atom will be in the excited state is

$$
P_2(N\tau) = \left(\frac{\alpha q r_{12}}{\hbar}\right)^2 \frac{\sin^2(\omega N\tau/2)}{\sin^2(\omega_0 \tau/2)}. \quad (29.2.5)
$$

We expand this when $\omega_0 \tau << 1$, and then also assuming $\omega_0 N\tau << 1$:

$$
P_2(N\tau) \approx \left(\frac{\alpha q r_{12}}{\hbar}\right)^2 \frac{\sin^2(\omega N\tau/2)}{(\omega_0 \tau/2)^2} \approx \left(\frac{Na q r_{12}}{\hbar}\right)^2. \quad (29.2.6)
$$
29.3 Time-Perturbed 1D Harmonic Oscillator

Spring 2008 Modern (Morning), Problem 2: A simple harmonic oscillator with angular frequency \( \omega \) containing a particle of mass \( m \) is in the \( n = 10 \) state, \( |10\rangle \), at time \( t_0 \) in the far distant past. A time-dependent perturbation \( H'(x, t) = ax^5e^{-(t/\tau)^2} \) is applied to the oscillator. Consider only the effects that are first order in \( a \).

(a) Write down all the possible values of \( n \) to which the state can transition.

(b) Calculate the probability that the oscillator will be in the final state from part (a) with the highest energy after a time \( t >> \tau \) has elapsed.

**SOLUTION:**

(a) We are looking for \( |n\rangle \) such that \( \langle n|x^5|10\rangle \neq 0 \). The energy is \( \frac{1}{2}m\omega^2x^2 + \frac{p^2}{2m} \). The natural energy unit is \( \hbar\omega \). Thus, in natural units, \( \frac{1}{2}m\omega^2x^2 + \frac{p^2}{2m} = (2m\hbar\omega)^{-1/2}(m\omega x + ip)(m\omega x - ip) \). Let \( a_\pm = (2m\hbar\omega)^{-1/2}(m\omega x \pm ip) \) so that \( x = (\hbar/2m\omega)^{1/2}(a_+ + a_-) \). The diagram below shows states connected to \( |10\rangle \) by \( 5x \)'s.

![Diagram showing states connected to |10⟩ by 5x's](image)

Figure 29.1: Spring 2008 Modern (Morning), Problem 2

\[
\int_{-\infty}^{\infty} f(x) \, dx = 0 \text{ if } f(x) \text{ is odd. } x^5 \text{ is odd, whereas } \langle x|10\rangle \text{ is even. Thus, we need odd } n. \]

Allowed: \( n = 5, 7, 9, 11, 13, 15 \).

(b) \( (\hbar/2m\omega)^{5/2}a_+^5 \) links \( |15\rangle \) to \( |10\rangle \). Recall that \( a_+ |n\rangle = \sqrt{n+1} |n+1\rangle \). Thus,

\[
\langle 15|x^5|10\rangle = (\hbar/2m\omega)^{5/2}\langle 15|a_+^5|10\rangle = (\hbar/2m\omega)^{5/2}(\frac{15}{10})^{1/2}. \quad (29.3.1)
\]

Write \( |\psi(t)\rangle = \sum_m c_m(t)e^{-i\omega_m t} |m\rangle \), where \( \omega_m = \omega (m + \frac{1}{2}) \). Then,

\[
H |\psi(t)\rangle = i\hbar \frac{d}{dt} |\psi(t)\rangle = \sum_m (i\hbar \dot{c}_m + \hbar \omega_m c_m)e^{-i\omega_m t} |m\rangle. \quad (29.3.2)
\]

On the other hand,

\[
(H_0 + H') |\psi(t)\rangle = \sum_m (E_m + H')c_m e^{-i\omega_m t} |m\rangle. \quad (29.3.3)
\]

Set these equal and cancel \( \hbar \omega_m \) with \( E_m \). Take the inner product with \( \langle n| \) to get

\[
\langle n|H'|m\rangle = \sum_m c_m e^{-i\omega_m t} \langle n|H'|m\rangle. \quad (29.3.4)
\]
29.3. TIME-PERTURBED 1D HARMONIC OSCILLATOR

Define $\omega_{nm} \equiv \omega_n - \omega_m = \omega(n - m)$ and $H'_{nm} \equiv \langle n|H'|m \rangle$ so that the rate equation may be written as

$$\dot{c}_n = -\frac{i}{\hbar} \sum_m c_{m,n}^{\omega_{nm}} H'_{nm}.$$  \hfill (29.3.5)

Plug in the initial conditions $c_m^{(0)} = \delta_{m,10}$ in the RHS. Then,

$$\dot{c}_{15} = -\frac{i}{\hbar} e^{i\omega(15-10)t} H'_{15,10} = -\frac{i\omega}{\hbar} e^{-(t/\tau)^2 + i5\omega t} (\hbar/2m\omega)^{5/2} (\frac{15}{10})^{1/2}. \hfill (29.3.6)$$

For $t >> \tau$, we may integrate all the way up to $\infty$:

$$c_{15} \approx -\frac{i\omega}{\hbar} \left( \frac{\hbar}{2m\omega} \right)^{5/2} \left( \frac{15!}{10!} \right)^{1/2} \int_0^\infty \exp\left\{ -\frac{1}{\tau^2} \left[ (t')^2 - i5\omega t'^2 - \left( \frac{5\omega\tau^2}{2} \right)^2 \right] - \left( \frac{5\omega\tau^2}{2} \right)^2 \right\}$$

$$= -\frac{i\omega\tau}{\hbar} \left( \frac{\hbar}{2m\omega} \right)^{5/2} \left( \frac{15!}{10!} \right)^{1/2} e^{-25\omega^2\tau^2/4} \int_0^\infty e^{-\xi^2} d\xi$$

$$= -\frac{i\omega\tau}{\hbar} \left( \frac{\hbar}{2m\omega} \right)^{5/2} \left( \frac{15!}{10!} \right)^{1/2} \sqrt{\frac{\pi}{2}} e^{-25\omega^2\tau^2/4}. \hfill (29.3.7)$$

Therefore, the probability of ending up in $|15\rangle$ is

$$P_{15} = |c_{15}|^2 = \frac{\pi}{4} \left( \frac{15!}{10!} \right) \left( \frac{\alpha}{\hbar} \right)^2 \left( \frac{\hbar}{2m\omega} \right)^{5/2} e^{-25\omega^2\tau^2/2}. \hfill (29.3.8)$$

Note: Professor’s solution is 4 times this!
29.4 Time-Perturbed 2D Harmonic Oscillator

[Spring 2004 Modern (Afternoon), Problem 4] Consider a two-dimensional isotropic harmonic oscillator of frequency $\omega = \sqrt{k/m}$ with a perturbing potential $\delta V(x, y) = bx y$, $b << k$, where $b$ has time dependence $b = b_0 e^{-t/\tau}$. Find to lowest order in $b_0$, an expression for the probability that a state that is in the ground state at $t = 0$ will be found in an excited state at $t = +\infty$.

**SOLUTION:**

Expand the wavefunction $|\psi(t)\rangle$ in energy eigenstates of the unperturbed Hamiltonian, which are labelled by two integers: $|m, n\rangle$,

$$|\psi(t)\rangle = \sum_{m', n'} c_{m', n'}(t) e^{-i\omega_{m', n'} t} |m', n'\rangle,$$

(29.4.1)

where $\omega_{m', n'} = E_{m', n'}/\hbar$.

Plugging this into Schrödinger’s equation and taking the inner product with $\langle m, n |$ gives the standard rate equation

$$\dot{c}_{m, n} = -\frac{i}{\hbar} \sum_{m', n'} \delta V_{m, n; m', n'} e^{i\omega_{m, n; m', n'}} c_{m', n'}(t),$$

(29.4.2)

where $\delta V_{m, n; m', n'} \equiv \langle m, n | \delta V | m', n'\rangle$ and $\omega_{m, n; m', n'} = \omega_{m, n} - \omega_{m', n'}$.

The zeroth order coefficients are $c_{m, n}^{(0)}(t) = \delta_{m, n; 0, 0}$ and so the first order corrected coefficients are

$$c_{m, n}^{(1)} = \delta_{m, n; 0, 0} e^{i\omega_{m, n; 0, 0} t}.$$

(29.4.3)

We have already written this perturbation in terms of raising and lowering operators in Problem 27.3: $\delta V = \frac{b\hbar}{2m\omega} (a_x^2 + a_x^1) (a_y^1 + a_y)$. Therefore, it connects states that are separated in by $|\pm 1, \pm 1\rangle$. Therefore, the only state that links with the ground state is $|11\rangle$. That is, $\delta V_{m, n; 0, 0} = \frac{b\hbar}{2m\omega} \delta_{m, n; 1, 1} e^{-t/\tau}$ and $\omega_{1, 1; 0, 0} = 2\omega$. Thus,

$$c_{m, n}^{(1)} = -\frac{i b_0}{2m\omega} \delta_{m, n; 1, 1} e^{i(2i\omega - \tau^{-1})t}.$$

(29.4.4)

With the initial condition $c_{m, n}(0) = \delta_{m, n; 0, 0}$, we get

$$c_{m, n}^{(1)}(t) = \delta_{m, n; 0, 0} - \frac{i b_0}{2m\omega} \delta_{m, n; 1, 1} \int_0^t dt' e^{(2i\omega - \tau^{-1})t'} = \delta_{m, n; 0, 0} - \frac{i b_0 \delta_{m, n; 1, 1}}{2m\omega} \frac{1 - e^{(2i\omega - \tau^{-1})t}}{\tau^{-1} - 2i\omega}.$$

(29.4.5)

Hence, the probability of being in an excited state is

$$|c_{1, 1}^{(1)}(\infty)|^2 = \frac{b_0^2}{4m^2\omega^2(4\omega^2 + \tau^{-2})}.$$

(29.4.6)
29.5 Harmonic Oscillator in Electric Field

[Kevin G.] A particle of mass \( m \) and electric charge \( q \) moves in one dimension in a harmonic potential. At \( t = 0 \) we turn on a homogeneous time-varying electric field.

(a) Write down the Hamiltonian for this system for all times.

(b) Exactly solve the energy eigenvalue problem for all times and relate the eigenstates of this system before and after the introduction of the field.

(c) If the system starts out in the ground state for \( t < 0 \), what is the probability that it will be in an excited state for \( t > 0 \)? You may use Zassenhaus’ formula:

\[
e^{\alpha(A+B)} = e^{\alpha A} e^{-\frac{\alpha^2}{2} [A,B]} e^{\frac{\alpha^2}{2} (2[B,[A,B]]+[A,[A,B]])} \ldots,
\]

where the dots indicate more and more nested commutators.

(d) Now, solve the problem using time-dependent perturbation theory treating the bare SHO as the unperturbed Hamiltonian. Do this for general \( \mathcal{E}(t) \), then specifically for \( \mathcal{E}(t) = \mathcal{E}_0 e^{-t/\tau} \).

SOLUTION:

(a) \[ H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2 - q \mathcal{E} x \theta(t) \]

where \( \theta(t) \) is the Heaviside step function, which vanishes for \( t < 0 \) and is 1 for \( t > 0 \).

(b) For \( t < 0 \), the eigenstates are \( |n\rangle \) and \( H = \hbar \omega (n + \frac{1}{2}) \), where \( n \in \mathbb{Z}_{\geq 0} \).

For \( t > 0 \), we may write \( H' = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 (x - \frac{q \mathcal{E}}{m \omega^2})^2 - \frac{q^2 \mathcal{E}^2}{2m \omega^2} \). Define the new position operator \( x' = x - \xi \) where \( \xi = \frac{q \mathcal{E}}{m \omega^2} \), then \( H' \) is just a displaced harmonic oscillator Hamiltonian in \( x' \).

Recall that, in momentum space, \( x = i \hbar \partial_p \), so that the exponential of \( p \) serves as a translation operator:

\[
e^{i p / \hbar} x e^{-i p / \hbar} = e^{i p / \hbar} (i \hbar \partial_p) e^{-i p / \hbar} = i \hbar (\partial_p - i \xi / \hbar) = x + \xi.
\]  

So, we may actually write \( H' = e^{-i p / \hbar} H e^{i p / \hbar} - \frac{1}{2} m \omega^2 \xi^2 \). Thus, it is clear that the new eigenstates are related to the old ones via \( |n\rangle' = e^{-i p / \hbar} |n\rangle \) and the new Hamiltonian is \( H' = \hbar \omega (n + \frac{1}{2}) - \frac{1}{2} m \omega^2 \xi^2 \). In terms of space wavefunctions, \( \psi_n'(x) = \psi_n(x - \xi) \).

(c) Let us compute the probability of remaining in the ground state (of course, the ground state itself changes by a translation). This is \( P_0 = |\langle 0 | 0 \rangle|^2 = |\langle 0 | 0 \rangle'|^2 = |\langle 0 | e^{-i p / \hbar} | 0 \rangle|^2 \).

**Method 1:** In terms of the raising and lowering operators for the bare harmonic oscillator, \( p = i \sqrt{\hbar m \omega / 2} (a^\dagger - a) \) and thus, \( e^{-i p / \hbar} = e^{\alpha (a^\dagger - a)} \), where \( \alpha = \sqrt{m \omega \xi^2 / 2 \hbar} \). Using Zassenhaus’ formula, all terms of order \( \geq \alpha^3 \) vanish since \( [a, a^\dagger] = 1 \) and so nested commutators all vanish. The result is

\[
e^{-i p / \hbar} = e^{\alpha a^\dagger} e^{-\alpha a} e^{-\alpha^2 / 2}.
\]  

The only term in the exponential that survives the expectation value in the ground state is the zeroth term (i.e. 1). Thus,

\[
P_0 = e^{-\alpha^2} = \exp \left[ -\frac{q^2 \mathcal{E}^2}{2m \omega^3 \hbar} \right].
\]  

(29.5.3)
The probability of being in an excited state is

$$P_e = 1 - P_0 = 1 - \exp\left[\frac{-q^2 \mathcal{E}^2}{2m\omega^2\hbar}\right]$$  \hspace{1cm} (29.5.4) \hspace{1cm} (\text{CHPT})

**Method 2**: Define $P_n = |\langle n | 0 \rangle'|^2$. Since $|n\rangle = \frac{1}{\sqrt{n!}} (a^\dagger)^n |0\rangle$ and $a^\dagger$ is the adjoint of $a$, we have $\langle n | 0 \rangle' = \frac{1}{\sqrt{n!}} (a^n |0\rangle |0\rangle'$. We may express $a$ in terms of the new raising operator $a'$ via

$$a = (2\hbar m\omega)^{-1/2}(m\omega x + ip) = (2\hbar m\omega)^{-1/2}(m\omega(x' + \xi) + ip') = a' + \alpha.$$  \hspace{1cm} (29.5.5)

Then, $a^n |0\rangle' = (a' + \alpha)^n |0\rangle' = \alpha^n |0\rangle'$, so that $\langle n | 0 \rangle' = \frac{\alpha^n}{\sqrt{n!}} \langle 0 | 0 \rangle'$. It follows that $P_n = e^{-\alpha^2}$, which agrees with Method 1.

**Method 3**: Recall that the ground state is just a Gaussian of unit width in the variable $\sqrt{m\omega/\hbar} x$. Normalizing it gives $\psi_0(x) = (m\omega/\pi\hbar)^{1/4} e^{-m\omega x^2/2\hbar}$. From part (b), $\psi_0'(x) = \psi_0(x - \xi)$. Therefore,

$$\psi_0'(0) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \int_{-\infty}^{\infty} e^{-m\omega/2\hbar}[x^2 - (x - \xi)^2] dx = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} e^{-m\omega \xi^2/2\hbar} \int_{-\infty}^{\infty} e^{-(m\omega/\hbar)(x - \frac{1}{2})^2} dx = e^{-m\omega \xi^2/2\hbar} \exp(\frac{q^2 \mathcal{E}^2}{4m\omega^3\hbar}).$$  \hspace{1cm} (29.5.6)

When squared, this gives the same $P_0$ as (29.5.3) and thus the same $P_e$ as (29.5.4).

(d) We expand the wavefunction in the bare SHO eigenstates $\psi(t) = \sum_n c_n(t)e^{-iE_n t/\hbar} |n\rangle$. The evolution of the coefficients is $\dot{c}_m = -\frac{i}{\hbar} \sum_n H_m^n e^{i\omega_m n t} c_n(t)$, where $H_m^n = -q \mathcal{E} (m|x|n)$ and $\omega_m = (m - n)\omega$.

The first order approximation is given by plugging in $c_n = \delta_{n0}$ to get $\dot{c}_n = \frac{i}{\hbar} q \mathcal{E} e^{i\omega t} \langle n | 0 \rangle$. Write $x = \sqrt{\hbar/2m\omega}(a^\dagger + a)$ to show that $\langle n | x | 0 \rangle = \sqrt{\hbar/2m\omega} \delta_{n0}$. Hence,

$$\dot{c}_1 = \sqrt{-q^2/2m\omega\hbar} \mathcal{E} e^{i\omega t}.$$  \hspace{1cm} (29.5.7)

Integrating this gives

$$c_1 = \sqrt{-\frac{q^2}{2m\omega\hbar}} \int_{-\infty}^{t} \mathcal{E}(t') e^{i\omega t'} dt',$$  \hspace{1cm} (29.5.8)

where it is understood that $\mathcal{E}(t < 0) = 0$.

As $t \to \infty$, this essentially gives the Fourier transform of $\mathcal{E}$:

$$c_1 = \sqrt{-\frac{\pi q^2}{m\omega\hbar}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathcal{E}(t) e^{i\omega t} dt = \sqrt{-\frac{\pi q^2}{m\omega\hbar}} \mathcal{E}(\omega).$$  \hspace{1cm} (29.5.9)

Therefore, the probability to be in an excited state is

$$P_e = |c_1|^2 = \frac{q^2}{2m\omega\hbar} \left| \int_{-\infty}^{t} \mathcal{E}(t') e^{i\omega t'} dt' \right|^2 \xrightarrow{t \to \infty} \frac{\pi q^2}{m\omega\hbar} |\mathcal{E}(\omega)|^2.$$  \hspace{1cm} (29.5.10)
In the case $E(t) = E_0 e^{-t/\tau}$, we get

$$P_e = \frac{q^2 E_0^2}{2m\omega^3 \hbar} \frac{1 - 2e^{-t/\tau} \cos(\omega t) + e^{-2t/\tau}}{1 + (\omega \tau)^{-2}} \xrightarrow{t \to \infty} \frac{q^2 E_0^2}{2m\omega^3 \hbar} \cdot \frac{1}{1 + (\omega \tau)^{-2}}.$$  \hfill (29.5.11)

Note that if it happens that $q^2 E/2m\omega^2 \hbar << 1$, then the exact result, (29.5.4), becomes

$$P_e \approx \frac{q^2 E_0^2}{2m\omega^3 \hbar} = \frac{q^2 E_0^2}{2m\omega^3 \hbar} e^{-2t/\tau}.$$  \hfill (29.5.12)

In particular, the two results don’t quite have the same $t \to \infty$ behavior. The exact answer vanishes in this limit, whereas the approximate answer approaches a constant, which is presumably small if $\omega \tau << 1$, meaning that the field decays at a much faster rate than the characteristic SHO oscillation.

Aside: Convince yourself that in the exact method, all higher excited states are accessible to the system, with less and less probability as $n$ increases. In the time-dependent perturbation approximation, only the first excited state was accessible. This is the only the case to first order. If we now plug in the first order form of $c_n(t)$ into the evolution equation, we generate a nonzero second order correction to $c_2$, and the process continues.
29.6 Selection Rules

(a) Counting spin, there are \(2n^2\) energy eigenstates in hydrogen with principal quantum number \(n\). Sketch the energy levels for \(n = 1, 2, 3\), including spin-orbit and other fine structure effects. Arrange the energy levels in columns, labelled by \(\ell\) value. Draw the energy levels as short horizontal lines in each column, but if there are several states of a given angular momentum and energy, just draw one line. Label the energy level with its quantum numbers (but not magnetic quantum numbers). The drawing need not be to scale, but it should show the ordering of the energy levels and give some qualitative idea of their relative spacing.

(b) Sketch in the allowed electric dipole transitions between levels of different \(n\). State the selection rules you are using.

(c) Derive the Clebsch-Gordan coefficients required to couple the orbital and spin angular momentum for a \(p\) state.

(d) Let \(R_n(\ell)\) be the radial wave functions in hydrogen, let \(Y_l^m(\Omega)\) be the angular wave functions, and let \(\alpha = \binom{1}{0}\) and \(\beta = \binom{n}{1}\) be the spin wave functions. Write the full wave functions for all four \(n = 2\) states of hydrogen in terms of these.

SOLUTION:

(a) The thing to keep in mind here is that fine structure breaks the degeneracy in \(\ell\), but preserves the degeneracy in \(j\). Thus, for example, one can get \(j = 1/2\) from \((\ell, s) = (0, 1/2)\) as well as from \((1, 1/2)\) and those states are degenerate. Furthermore, fine structure decreases the energy \(E_n = E_1/n^2\) with the lowest \(j\) being decreased the most. In the diagram below, the dotted lines indicate the Bohr energy levels.

(b) \(\Delta \ell = \pm 1\) and \(\Delta j = 0, \pm 1\) (no \(3d_{5/2} \to 2p_{1/2}\) transition).

(c) Recall \(J_\pm \ket{j, m_j} = \sqrt{j(j + 1) - m_j(m_j - 1)} \ket{j, m_j - 1}\). The highest weight state is \(\ket{3/2, 3/2} = \ket{1} \ket{1/2}\), where the latter reads \(\ket{j, m_j} = \ket{m_\ell} \ket{m_s}\). Lowering once reads

\[
\sqrt{\frac{3}{2} \left(\frac{5}{2} - \frac{3}{2} \right)} \ket{\frac{3}{2}, \frac{1}{2}} = \sqrt{1(2) - 1(0)} \ket{\frac{1}{2}} + \sqrt{\frac{1}{2} \left(\frac{3}{2} - \frac{1}{2} \right)} \ket{1} - \frac{1}{2}
\]

\[
\ket{\frac{3}{2}, \frac{1}{2}} = \sqrt{\frac{3}{2}} \ket{0} \ket{\frac{1}{2}} + \sqrt{\frac{1}{2}} \ket{1} \ket{-\frac{1}{2}}.
\]
Lowering two more times gives
\[ |\frac{3}{2}, -\frac{1}{2}\rangle = \sqrt{\frac{5}{3}} |\frac{1}{2}, -\frac{1}{2}\rangle + \sqrt{\frac{2}{3}} |0, \frac{3}{2}\rangle, \quad |\frac{3}{2}, -\frac{3}{2}\rangle = |\frac{1}{2}, -\frac{1}{2}\rangle \cdot \quad (29.6.2) \]

The state $|1/2, 1/2\rangle$ must be a linear combination of $|1\rangle | -1/2\rangle$ and $|0\rangle |1/2\rangle$. Requiring that it be orthogonal to $|3/2, 1/2\rangle$ fixes the coefficients. Then, we lower once to get $|1/2, -1/2\rangle$. The result is
\[ |\frac{1}{2}, \frac{1}{2}\rangle = -\sqrt{\frac{1}{3}} |0\rangle |\frac{1}{2}\rangle + \sqrt{\frac{2}{3}} |1\rangle |\frac{1}{2}\rangle, \quad (29.6.3) \]
\[ |\frac{1}{2}, -\frac{1}{2}\rangle = -\sqrt{\frac{2}{3}} |0\rangle |\frac{1}{2}\rangle + \sqrt{\frac{1}{3}} |0\rangle |\frac{1}{2}\rangle. \]

(d) Using the Clebsch-Gordan coefficients from part (c),

<table>
<thead>
<tr>
<th>Spect.</th>
<th>$\ell$</th>
<th>$j$</th>
<th>$m_j$</th>
<th>Wave Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2s_{1/2}$</td>
<td>0</td>
<td>1/2</td>
<td>1/2</td>
<td>$R_{20}Y_0^0 \alpha$</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>1/2</td>
<td>-1/2</td>
<td>$R_{20}Y_0^0 \beta$</td>
</tr>
<tr>
<td>$2p_{1/2}$</td>
<td>1</td>
<td>1/2</td>
<td>1/2</td>
<td>$R_{21}(-\sqrt{\frac{1}{3}Y_1^0 \alpha + \sqrt{2/3}Y_1^1 \beta})$</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1/2</td>
<td>-1/2</td>
<td>$R_{21}(-\sqrt{2/3}Y_1^{-1} \alpha + \sqrt{1/3}Y_1^0 \beta)$</td>
</tr>
<tr>
<td>$2p_{3/2}$</td>
<td>1</td>
<td>3/2</td>
<td>3/2</td>
<td>$R_{21}Y_1^0 \alpha$</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>3/2</td>
<td>1/2</td>
<td>$R_{21}(\sqrt{2/3}Y_1^0 \alpha + \sqrt{1/3}Y_1^{-1} \beta)$</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>3/2</td>
<td>-1/2</td>
<td>$R_{21}(\sqrt{1/3}Y_1^{-1} \alpha + \sqrt{2/3}Y_1^0 \beta)$</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>3/2</td>
<td>-3/2</td>
<td>$R_{21}Y_1^{-1} \beta$</td>
</tr>
</tbody>
</table>
29.7 Rate of Ionization of Hydrogen

A hydrogen atom in its ground state sits at the origin. An electromagnetic wave, described by the vector potential \( A(r, t) = A_0 e^{i(k_0 \cdot r - \omega t)} \), in incident upon the atom. What is the rate of ionization of the hydrogen atom? [Assume: (1) the dipole approximation; and (2) the outgoing electron is in a plane wave state (i.e. ignore the Coulomb interaction with the proton once the electron is liberated); incidentally, this assumption is good if the initial state is an \( s \)-state, as it is here.]

**SOLUTION:**

The interaction between the atom and the EM wave is introduced by the slight modification of the momentum operator. Now, the *canonical* momentum conjugate to position is not \( p \), but \( p - \frac{\hbar}{c} \mathbf{A} = p + \frac{\hbar}{c} \mathbf{A} \). For those of you familiar with field theory, since momentum in position space is a derivative, this procedure is precisely equivalent to covariantizing the derivative \( \partial^\mu \rightarrow \partial^\mu - \frac{i}{\hbar} A^\mu \), which is precisely how photons couple to anything that has charge. Thus, the Hamiltonian is

\[
H = H_0 + \frac{e}{mc} (\mathbf{p} \cdot \mathbf{A} + \mathbf{A} \cdot \mathbf{p}) + \mathcal{O}(A^2),
\]

where \( H_0 \) is the usual hydrogen Hamiltonian in the absence of the EM wave and where we have expanded the term \((p + \hbar A)^2/2m\) and kept the \(p^2/2m\) term in \(H_0\) and the rest as perturbations. We will neglect terms of order \(A^2\), though, as it turns out, they have no effect on this problem anyway.

Since \( \nabla \cdot \mathbf{A} = 0 \) (Coulomb gauge), we may write the Hamiltonian as

\[
H = H_0 + \frac{e}{mc} \mathbf{A} \cdot \mathbf{p},
\]

where one has to keep in mind that \( \mathbf{p} \) and \( \mathbf{A} \) are operators and thus in the operator \( \mathbf{p} \cdot \mathbf{A} \), the derivative may act on \( \mathbf{A} \) itself, which vanishes by the Coulomb gauge, OR on the wavefunction on which this whole operator is acting from the left. Note that this gauge is equivalent to \( k_0 \cdot A_0 \), or, thus, that the plane wave is *transverse*, as EM waves ought to be. Thus, the perturbation to the atomic Hamiltonian is

\[
H'(t) = H' e^{-i\omega t} \quad \text{where} \quad H' = \frac{e}{mc} e^{ik_0 \cdot r} A_0 \cdot \mathbf{p}.
\]

We will need to compute the expectation value of \( H' \) between the hydrogen ground state and a plane wave outgoing state.
Scattering

- Green's function for $\nabla^2 + k^2$, where $k = \sqrt{2mE/\hbar}$ may be written $G(r) = -\frac{e^{ikr}}{4\pi r}$.

- Integral version of S.E.: $\psi(r) = \psi_0(r) - \frac{m}{2\pi\hbar^2} \int \frac{e^{ikr' - kr}}{|r - r'|} V(r') \psi(r') d^3 r'$.

- Radiation zone for scattering: $\psi(r) = Ae^{ikz} - \frac{m}{2\pi\hbar^2} \frac{e^{ikr}}{r} \int e^{-ikr'} V(r') \psi(r') d^3 r'$.

- Zeroth order: $\psi^{(0)}(r) = Ae^{ikz} = Ae^{ik' \cdot r'}$, where $k' = k \hat{z}$.

- First order: plug in $\psi^{(0)}$ on the RHS of the expression for $\psi$. Then, we find $\psi^{(1)} - \psi^{(0)} = -\frac{mA}{2\pi\hbar^2} \frac{e^{ikr}}{r} \int e^{-i(k - k') \cdot r'} V(r') d^3 r' = -\frac{(2\pi)^{3/2} mA}{\hbar^2} \frac{e^{ikr}}{r} V(q)$, where $q = k - k'$ is the momentum transfer and $V(q)$ is the Fourier transform of $V(r)$.

- Scattering Amplitude: $\psi = A(\psi_0 + f(\theta) \frac{e^{ikr}}{r})$.

- First Born approximation: Hence, $f(\theta, \phi) = -\frac{(2\pi)^{1/2} mA}{\hbar^2} V(q)$.

- Optical theorem: $\text{Im} f(0) = k\sigma/4\pi$.

- Differential cross section: $\frac{d\sigma}{d\Omega} = |f(\theta, \phi)|^2$.

- Partial waves: for s-wave scattering ($\ell = 0$; low-energy) $\sigma \approx \frac{4\pi}{k^2} \sin^2 \delta_0$ where the $u$-wavefunction outside the scattering potential range is $u(r) \sim \sin(kr + \delta_0)$. 
30.1 Spherical Step Potential Scatterer

[Kevin G.] Compare the partial waves and the Born approximation scattering cross sections for a low energy particle from a potential given by \( V = -V_0 \) for \( r < a \) and \( V = 0 \) for \( r > a \).

**SOLUTION:**

**Partial Waves:** Since the incoming particles have low energy, their scattering will be dominated by the \( s \) partial wave (\( l = 0 \)). In this case, the radial Schrödinger equation reads \( \left[ \frac{d^2}{dr^2} + k^2 \right] u(r) = 0 \) for \( r > a \), where \( k^2 = 2mE/\hbar^2 \), and similarly \( k \to k' \), where \( k'^2 = 2m(E + V_0)/\hbar^2 \) for \( r < a \). The requirement that \( u \to 0 \) as \( r \to \infty \) (so that \( R(r) = u(r)/r \) is finite) implies \( u(r) = A \sin(k'r) \) for \( r < a \). A general solution for the outside region is \( u(r) = B \sin(kr + \delta_0) \).

Continuity of \( u \) and \( u' \) at \( r = a \) implies the relation \( k \tan(k'a) = k' \tan(ka + \delta_0) \), which can be solved for \( \delta_0 \) to give

\[
\delta_0 = \tan^{-1}\left[ \frac{k}{k'} \tan(k'a) \right] - ka. \tag{30.1.1}
\]

For low energy scattering, \( k \to 0 \) and \( k' \to k_0 \equiv \sqrt{2mV_0}/\hbar \). Then, we get

\[
\delta_0 \approx ka \left[ \frac{\tan(k_0a)}{k_0a} - 1 \right]. \tag{30.1.2}
\]

The total scattering cross-section is

\[
\sigma \approx \frac{4\pi}{k^2} \sin^2 \delta_0 \approx \frac{4\pi}{k^2} \delta_0^2 = 4\pi a^2 \left[ \frac{\tan(k_0a)}{k_0a} - 1 \right]^2. \tag{30.1.3}
\]

If \( k_0a << 1 \) (i.e. the interaction is short-ranged), then

\[
\sigma \approx 4\pi a^2 \left[ \frac{(k_0a)^2}{3} + \mathcal{O}(k_0a)^4 \right]^2 = 4\pi \left( \frac{2ma^3V_0}{3\hbar^2} \right)^2. \tag{30.1.4}
\]

**Born Approximation:** In the first Born approximation,

\[
V(q) = -\frac{2\pi V_0}{(2\pi)^{3/2}} \int_0^a dr r^2 \int_{-1}^1 e^{-iqrx} dx
\]

\[
= -\frac{iV_0}{(2\pi)^{1/2}q} \int_0^a dr (e^{iqr} - e^{-iqr})
\]

\[
= -\frac{2V_0}{(2\pi)^{1/2}q^3} \left[ \sin(qa) - qa \cos(qa) \right]. \tag{30.1.5}
\]

Thus, the scattering amplitude is

\[
f(\theta) = -\left( \frac{2\pi}{\hbar} \right)^{1/2} \frac{mV_0}{k^2q^3} \left[ \sin(qa) - qa \cos(qa) \right]. \tag{30.1.6}
\]

For low energies and short range, \( qa \to 0 \), and so

\[
f(\theta) \approx \frac{2ma^3V_0}{3\hbar^2}. \tag{30.1.7}
\]

The differential cross section is \( d\sigma/d\Omega = |f(\theta)|^2 \) is constant and thus the total cross section is simply this multiplied by \( 4\pi \):

\[
\sigma = 4\pi \left( \frac{2ma^3V_0}{3\hbar^2} \right)^2, \tag{30.1.8}
\]
30.1. SPHERICAL STEP POTENTIAL SCATTERER

in agreement with the partial waves analysis.

**Fermi’s Golden Rule:** We box normalize in a volume $V$, so that the incoming wave is $\psi_i(\mathbf{r}) = \frac{1}{\sqrt{V}} e^{i \mathbf{k} \cdot \mathbf{r}}$ and the outgoing wave is $\psi_f(\mathbf{r}) = \frac{1}{\sqrt{V}} e^{i \mathbf{k}' \cdot \mathbf{r}}$. Then,

$$\langle f | V(\mathbf{r}) | i \rangle = \frac{1}{V} \int_V d^3 r e^{-i (\mathbf{k}' - \mathbf{k}) \cdot \mathbf{r}} = \frac{4\pi V_0}{V q^3} [\sin(qa) - qa \cos(qa)],$$

(30.1.9)

since the only difference between this and (30.1.5) is a factor of $(2\pi)^{3/2}/V$. In the $qa << 1$ limit, this gives

$$\langle f | V(\mathbf{r}) | i \rangle \approx \frac{4\pi a^3 V_0}{3V}.$$  

(30.1.10)

Using the same density of states as in Problem 30.2, we find the transition rate

$$w = \frac{2\pi}{\hbar} \left( \frac{4\pi a^3 V_0}{3V} \right)^2 \frac{4\pi m p_i V}{(2\pi \hbar)^3} = \frac{4\pi p_i}{mV} \left( \frac{2ma^3 V_0}{3\hbar^2} \right)^2.$$  

(30.1.11)

The rate and cross section are related via $dw = n v_i d\sigma = (p_i / mV) d\sigma$ (c.f. (30.2.10)) and, since $w$ is independent of angle, we can just write $w = (p_i / mV) \sigma$. This gives

$$\sigma = 4\pi \left( \frac{2ma^3 V_0}{3\hbar^2} \right)^2,$$

(30.1.12)

which agrees with the previous two methods.
CHAPTER 30. SCATTERING

30.2 Delta Function Scatterer

[Fall 2007 Modern (Morning), Problem 5] Calculate, to within a multiplicative constant, the differential cross section for scattering of particles of momentum $p$ in a potential which can be represented by a delta function, $V(r) = \frac{a}{\sqrt{2\pi a^2}} \delta^2(r)$, where $a$ is a constant. Use the Born approximation or time-dependent perturbation theory and the Fermi golden rule. Show that a similar energy and angular dependence is obtained for differential cross sections in any finite range of potential at “low” momentum. Give the condition on “low” momentum for this result to follow.

SOLUTION:

Method 1: Dealing with this rather singular potential is a bit tricky. Instead, I will approximate it by a Gaussian:

$$V(r) = \frac{a}{(2\pi)^{3/2}} e^{-s^2 q^2/2},$$
(30.2.1)

Thus, the scattering amplitude is

$$f(\theta) = -\frac{ma}{2\pi \hbar^2} e^{-s^2 q^2/2}.$$
(30.2.2)

Plugging in $q^2 = 4k^2 \sin^2(\theta/2)$, the differential cross section is

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \left( \frac{ma}{2\pi \hbar^2} \right)^2 e^{-4(k\hbar s)^2 \sin^2(\theta/2)}.$$  
(30.2.3)

We require that the wavelength of the scattering particles be much larger than the size of the scattering potential, and thus $ks << 1$. Hence, to first order,

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \left( \frac{ma}{2\pi \hbar^2} \right)^2.$$  
(30.2.4)

Let $s$ be the range of the potential. If $ks << 1$, as above, then the exponential in the Fourier transform is $\frac{1}{\pi}$ and so the Fourier transform is just the integral of the potential over all space, which is here defined to be $a$. Thus, we still get $f(\theta) \approx -ma/2\pi \hbar^2$ and everything follows as above.

Method 2: Let us box-normalize our wavefunctions in a volume $V$, so that the incoming wave is $\psi_i(r) = \frac{1}{\sqrt{V}} e^{i k^0 r}$ and the outgoing wave is $\psi_f(r) = \frac{1}{\sqrt{V}} e^{i k^0 r}$. Then,

$$\langle f | V(r) | i \rangle = \frac{1}{V} \int_V d^3r e^{-i(k-k^0)^2 r} V(r) = \frac{a}{V}.$$  
(30.2.5)

The transition rate is given by Fermi’s golden rule:

$$w = \frac{2\pi}{\hbar} |\langle f | V(r) | i \rangle|^2 \rho_f(E_i),$$
(30.2.6)

where $\rho_f(E_i)$ is the density of final states, $\rho_f(E) = dN_f/dE$, evaluated at the initial (incoming) energy.
As per the usual counting of states, \( N(k) = \frac{1}{8} \times \frac{1}{2} \times \frac{4}{3} \pi n(k)^3 \) since \( k = (\pi/L)n \). Thus, in terms of energy, we have \( N(E) = \frac{V}{6\pi^2} \left( \frac{2mE}{h^2} \right)^{3/2} \) and we get

\[
\rho(E) = \frac{dN}{dE} = V \left( \frac{2mE}{h^2} \right)^{3/2} E^{1/2} = \frac{4\pi mpV}{(2\pi\hbar)^3},
\]

where I just substituted in \( E = p^2/2m \) at the end.

Therefore, the transition rate is

\[
w = \frac{2\pi a^2}{\hbar} \frac{4\pi mpV}{(2\pi\hbar)^3} = \frac{4\pi p_i}{mV} \left( \frac{ma}{2\pi\hbar^2} \right)^2.
\]

Taking an angular differential simply divides by \( 4\pi \) since \( w \) is independent of angle:

\[
\frac{d\sigma}{d\Omega} = \frac{p_i}{mV} \left( \frac{ma}{2\pi\hbar^2} \right)^2.
\]

The rate and cross section are related via

\[
d\sigma = n v_i d\sigma = (p_i/mV) d\sigma,
\]

where \( n \) is the number of incident particles per volume, which for single scattering we will set equal to \( 1/V \), and \( v_i \) is the incoming speed. Note that it is important that the \( p_i \) that shows up in (30.2.10) is the same as the one that shows up in (30.2.9) since the latter contains the density of states evaluated at the incoming energy, or incoming momentum.

Therefore, we find

\[
\frac{d\sigma}{d\Omega} = \frac{mV}{p_i} \frac{dw}{d\Omega} = \left( \frac{ma}{2\pi\hbar^2} \right)^2.
\]

**Method 3:** Let \( V = V_0 \) for \( r < b \) and \( V = 0 \) for \( r > b \). We will take the \( b \to 0 \) limit while keeping \( a = \frac{3}{4}\pi b^3 V_0 \) constant. Low energy means \( s (= 0) \) partial wave scattering. In this case, the radial Schrödinger equation reads \( \frac{d^2}{dr^2} + k^2 \) \( u(r) = 0 \) for \( r > b \), where \( k^2 = 2mE/h^2 \).

Similarly for \( r < b \), but \( k \to k' \), where \( k'^2 = 2m(E - V_0)/h^2 \). \( R(r) = u(r)/r \) must be finite as \( r \to 0 \), so \( u(r) = A \sin(k'r) \) for \( r < b \). The general solution outside is \( u(r) = B \sin(kr + \delta_0) \).

Continuity of \( u \) and \( u' \) at \( r = b \) implies \( k \tan(k'b) = k' \tan(kb + \delta_0) \) whose solution is

\[
\delta_0 = \tan^{-1}\left( \frac{k}{k'} \tan(k'b) \right) - kb \approx kb \left( \frac{\tan(kb)}{k_0b} - 1 \right),
\]

where, for low energy, \( k \to 0 \) and \( k' \to k_0 \equiv \sqrt{2mV_0}/\hbar \).

In the short range limit, \( k_0b \ll 1 \) and so

\[
\delta_0 \approx \frac{kb}{k_0b} \left[ k_0b + \frac{2mV_0b^3 - k_0b}{3} \right] = \frac{kk_0^3b^3}{3} - k \frac{2mV_0b^3}{3h^2} = \frac{kma}{2\pi\hbar^2},
\]

where use was made of \( a = \frac{3}{4}\pi b^3 V_0 \).

This gives us the total cross section \( \sigma \approx \frac{4\pi}{k^2} \sin^2 \delta_0 \approx \frac{4\pi}{k^2} \delta_0^2 = 4\pi \left( \frac{ma}{2\pi\hbar^2} \right)^2 \). Since the scattering is isotropic, we need only divide by \( 4\pi \) to get the differential cross section, which gives the same answer as the previous methods.

Note that our final answer is independent of \( b \) and is unaltered by the \( b \to 0 \) limit! This clue us into the fact that for sufficiently low momentum, implying short potential range, the scattering is insensitive to the precise form of the potential. In this limit, \( V(q) \) is just the integral of \( V(r) \).
30.3 Yukawa Potential Scatterer

[Spring 2004 Modern (Afternoon), Problem 6]

(a) Use the first Born approximation to estimate the differential and total cross sections for the scattering of a particle by a Yukawa potential $V = e^{-\mu r}/r$.

(b) Use the optical theorem and the result of (a) to estimate the imaginary part of the forward scattering amplitude.

SOLUTION:

(a) We must first Fourier transform this potential:

$$
V(q) = \frac{1}{(2\pi)^{3/2}} \int_0^\infty dr \frac{r^2 g e^{-\mu r}}{r} \int_{-1}^1 e^{-iqr} dr \int_0^{2\pi} d\phi
$$

$$
= \frac{g}{(2\pi)^{1/2}} \int_0^\infty dr \frac{1}{iqr} \left[ e^{iqr} - e^{-iqr} \right]
$$

$$
= \frac{g}{(2\pi)^{1/2}iq} \int_0^\infty dr \left[ \frac{1}{\mu - iq} - \frac{1}{\mu + iq} \right] \int_0^\infty dx e^{-x}
$$

$$
= \frac{g}{(2\pi)^{1/2}iq} \left( \frac{2i}{\mu^2 + q^2} \right)
$$

Thus, the scattering amplitude is

$$
f(\theta) = -\frac{(2\pi)^{1/2}m}{\hbar^2} V(q) = -\frac{2mg}{\hbar^2(\mu^2 + q^2)}.
$$

Here, $q = |k - k'| = 2k^2(1 - \cos \theta) = 4k^2 \sin^2(\theta/2)$ and so the differential cross section is

$$
\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \frac{4m^2g^2}{\hbar^4[\mu^2 + 4k^2\sin^2(\theta/2)]^2}
$$

The total cross section is just the angular integral of this:

$$
\sigma = \frac{8\pi m^2g^2}{\hbar^4} \int_{-1}^1 \frac{dx}{\left( \mu^2 + 2k^2 - 2k^2x \right)^2} = \frac{16\pi m^2g^2}{\hbar^4\mu^2(\mu^2 + 4k^2)}
$$

(b) The optical theorem says

$$
\text{Im} \, f(0) = \frac{k\sigma}{4\pi} = \frac{4km^2g^2}{\hbar^4\mu^2(\mu^2 + 4k^2)}
$$
Part V

Quizzes
1. What is the Lagrangian of a double pendulum both pendulums of which have length $\ell$ and mass $m$.

**SOLUTION:** Let $\theta$ and $\phi$ be the angles of the top and bottom pendulums, respectively. The position of the top mass is $(x, y)_1 = \ell(\sin \theta, -\cos \theta)$ and of the bottom mass is $(x, y)_2 = \ell(\sin \theta + \sin \phi, -\cos \theta - \cos \phi)$. Then, $L = \frac{1}{2}m\ell^2 [2\ddot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}\cos(\theta - \phi) + 2(g/\ell)(2\cos \theta + \cos \phi)]$.

2. I measure my heartbeat at 60 bpm, but I’m also riding on a train that’s moving at $v = 3c/5$ left to right from your point of view. What do you measure my heartbeat to be?

**SOLUTION:** $60/\gamma = 60/(5/4) = 48$ bpm.

3. What is the expression for the centrifugal force?

**SOLUTION:** $-m\omega \times (\omega \times r)$.

4. Describe how to take account of a constraint in Lagrangian mechanics.

**SOLUTION:** Let the constraint be $g(q_i, \dot{q}_i) = 0$. Add a Lagrange multiplier to the Lagrangian: $L_{\text{tot}} = L + \lambda g$. Now, there are $N + 1$ unknowns: $q_i$ and $\lambda$. But, there are also $N + 1$ equations: $N$ E-L EoM and the constraint equation.

5. Relate torque and angular momentum.

**SOLUTION:** $\tau = \dot{L}$.

6. What is Bernoulli’s equation?

**SOLUTION:** $P + \frac{1}{2}\rho u^2 + \rho gh = \text{constant}$.

7. What is the formula for the eccentricity of a gravitational orbit and describe the possible orbit shapes.

**SOLUTION:** $\epsilon = \left(1 + \frac{2EL^2}{m^2r^2}\right)^{1/2}$, where the gravitational potential is $V(r) = -\alpha/r$. The orbits are: circle ($\epsilon = 0$; also where $V_{\text{eff}} = V + \frac{L^2}{2mr^2}$ has an extremum), ellipse ($0 < \epsilon < 1$), parabola ($\epsilon = 1$), and hyperbola ($\epsilon > 1$).

8. What are the normal mode frequencies of a spring-mass-spring-mass-spring system on a frictionless tabletop and attached to walls on either end, such that, at equilibrium, none of the springs are stretched or compressed and only longitudinal motion is allowed? The masses are $m$ and spring constants are $k$. 

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SOLUTION: Let $x_1$ and $x_2$ be the displacements of each mass. The equations are $m\ddot{x}_1 = k(x_2 - x_1) - kx_1 = k(-2x_1 + x_2)$ and $m\ddot{x}_2 = -kx_2 - k(x_2 - x_1) = k(x_1 - 2x_2)$. Plugging in the normal mode ansatz, $x_1 = A e^{i\omega t}$ and $x_2 = B e^{i\omega t}$, yields $(\omega^2 - 2\omega_0^2)A^2 = 0$. Setting the determinant to zero yields $(\omega^2 - 2\omega_0^2)^2 - \omega_0^4 = (\omega^2 - \omega_0^2)(\omega^2 - 3\omega_0^2) = 0$. Thus, the two normal mode frequencies are $\omega = \omega_0 = \sqrt{k/m}$ and $\omega = \sqrt{3}\omega_0 = \sqrt{3k/m}$.

9. Explain buoyancy.

SOLUTION: An object with mass density $\rho_{\text{obj}}$ is submerged in a fluid with mass density $\rho_{\text{fluid}}$ and displaces a certain volume, $V_{\text{disp}}$, of the fluid (i.e. that’s the volume that is submerged in the fluid). The buoyancy force is $F_{\text{bouy}} = (\rho_{\text{fluid}} - \rho_{\text{obj}})V_{\text{disp}}$ upwards.

10. What is the speed of sound in air?

SOLUTION: 340 m/s.

11. What are Kepler’s laws?

SOLUTION: (1) planets move in elliptical orbits with the sun at one focus; (2) the radius vector to a planet sweeps out area at a rate that is independent of its position in the orbit; (3) $T^2 = 4\pi^2a^3/GM_\odot$, where $T$ is the orbit period, $a$ is the semi-major axis and $M_\odot$ is the sun mass.


SOLUTION: I’ll estimate the plane speed to be $\sim 250$ m/s (around 550 mph). This order of magnitude should be reasonable since a fast car might go at 100 mph and the speed of sound in air is 340 m/s. The speed of the bird is negligible in comparison. Estimate the bird to have length 25 cm. Then, the impact time is roughly, $\Delta t \approx 0.25$ m/250 m/s = $10^{-3}$ s. In this time, the bird is accelerated from $\sim 0$ m/s to 250 m/s. Thus, the average acceleration is $\vec{a} \approx 250$ m/s/10$^{-3}$ s = $2.5 \times 10^5$ m/s$^2$. Let the bird have mass $m \approx 0.4$ kg, then the average force is $\vec{F} = m\vec{a} \approx 10^5$ N.

13. Qualitatively explain why most metals are ductile.

SOLUTION: Metallic bonding is not so strict and lattice-structured as the standard covalent or ionic bonding that occurs in most other solids. The delocalized electron sea screens what would otherwise be strong repulsive forces between the atoms and thus allows them to slide past each other in response to tensile stress.

14. What is the altitude (from sea level) of geosynchronous orbit around the earth?

SOLUTION: We will need to know the radius of the earth, which is roughly 6400 km. Let $\omega$ be the angular frequency and $r$ the radius of orbit, then the centripetal acceleration is $a_c = \omega^2 r$, which must also be $G_N M/r^2$, where $M$ is the mass of the earth. Solving for $r$ gives $r^3 = GM/\omega^2$, or $(r/R)^3 = g/R\omega^2$, where $R$ is the radius of the earth and $GM/R^2 = g = 9.8$ m/s$^2$ is acceleration due to gravity on Earth’s surface. Plugging in $\omega = 2\pi$ rad / 86,400 s = $7.27 \times 10^{-5}$ rad/s gives $r/R \approx 6.6$. Then, the altitude is $\sim 5.6R = 35,900$ km.

15. Write down the Coriolis force.

SOLUTION: $-2m\omega \times \vec{v}$.

16. Use sensible arguments to “derive” the speed of sound in a deformable solid in terms of whichever parameters you feel are appropriate.

SOLUTION: Sound is a longitudinal wave, so it must involve the elasticity of the solid, which measures how much it stretches and contracts relative to its equilibrium length (strain;
unitless) when a force is applied across some cross-sectional area (stress; \(N/m^2 = J/m^3\)). In this case, what we want is Young’s modulus, \(Y = \frac{\text{stress}}{\text{strain}}\) for longitudinal forces, as for sound. \(Y\) has units of pressure, or energy density. If we divide energy density by mass density, we get something with units of speed squared. Hence, \(v = \sqrt{Y/\rho}\), which is, in fact, correct; there are no numerical prefactors.

17. A small mass \(m\) rests at the edge of a horizontal disk of radius \(R\); the coefficient of static friction between the mass and the disk is \(\mu\). The disk is rotated about its axis at an angular velocity such that the mass slides off the disk and lands on the floor, \(h\) meters below. What was the mass’ horizontal distance of travel from the point that it left the disk?

**SOLUTION:** The horizontal speed of the mass right when it slides off is given by \(mv^2/R = \mu mg\), or \(v = \sqrt{\mu R g}\). The time it takes to fall a distance \(h\) is \(t = \sqrt{2h/g}\). Hence, the horizontal distance travelled in this time is \(vt = \sqrt{2\mu Rh}\).

18. What is the moment of inertia of a square around an axis passing through one of its corners and perpendicular to the plane of the square?

**SOLUTION:** Let the square have mass \(M\) and sidelength \(L\). Then, its mass density is \(\rho = M/L^2\). The mass of a bit of area is \(dm = \rho dx dy = \frac{M}{L^2} dx dy\). Let the axis pass through the origin and the square extend in \(x\) and \(y\) from 0 to \(L\). Then, \(I = \int r^2 dm = \frac{M}{L^2} \int_0^L (x^2 + y^2) dx dy = \frac{M}{L^2} 2L \int_0^L x^2 dx = \frac{1}{3} ML^2\).

19. What is the maximum horizontal range of a projectile with mass \(M\) and initial speed \(v\) (on flat ground)?

**SOLUTION:** The time it takes to reach the top of the trajectory can be found using \(v_f = v_i + at\) with \(v_i = v \sin \theta\), \(v_f = 0\) and \(a = -g\). This gives \(t = \frac{v}{g} \sin \theta\). The horizontal range is the horizontal speed, \(v \cos \theta\), multiplied by the time it takes to hit the ground again, which is \(2t\), so \(R = \frac{2v}{g} \sin \theta \cos \theta = \frac{v^2}{g} \sin 2\theta\). The maximum \(R\) is \(R_{\text{max}} = \frac{v^2}{g}\) (when \(\theta = 45^\circ\)).

20. Describe the no-slip rolling condition.

**SOLUTION:** The tangential (usually, though not always, linear) distance travelled by the center of mass of the rolling object is equal to the arclength distance that a point on the edge of the rolling object has travelled. In other words \(R \theta = \ell\).
Chapter 32

E & M Quiz Solutions

1. Relate the surface and volume bound charge densities to the relevant quantity(ies).
   \[ \sigma_b = P \cdot \hat{n} \text{ and } \rho_b = -\nabla \cdot P. \]

2. Relate $D$, $E$ and $P$.
   \[ \text{SOLUTION: } D = \varepsilon_0 E + P. \]

3. Define the atomic polarizability tensor.
   \[ \text{SOLUTION: } \alpha = \nabla E, \text{ where } \alpha \text{ is the atomic dipole moment.} \]

4. Relate the dielectric constant and index of refraction.
   \[ \text{SOLUTION: } n = \sqrt{\varepsilon}. \]

5. Express the force on a dielectric between the plates of a capacitor.
   \[ \text{SOLUTION: } F = \frac{1}{2} V^2 \frac{dC}{dx}, \text{ where } V \text{ is the voltage across the capacitor, } C \text{ is its capacitance,} \]
   \[ \text{and } x \text{ is the direction in which the dielectric is being pulled out.} \]

6. A dodecahedron has twelve regular-pentagonal faces. One face sits at potential $V$ and is
   insulated from the rest of the faces which are grounded. What is the potential, $V_c$, at the
   center?
   \[ \text{SOLUTION: Symmetry dictates that } V_c \text{ not depend on which face is not grounded. Super-} \]
   \[ \text{position implies that } 12V_c \text{ would be the potential if all faces were at } V. \text{ But, in this latter case,} \]
   \[ V \text{ would be the potential everywhere inside the shape since this constant potential obviously} \]
   \[ \text{satisfies } \nabla^2 V = 0 \text{ and the correct boundary condition. Hence, } 12V_c = V, \text{ or } V_c = V/12. \]

7. What is the energy density stored in an electric field in matter?
   \[ \text{SOLUTION: } u_E = \frac{1}{2} D \cdot E. \]

8. What is the field of a magnetic dipole?
   \[ \text{SOLUTION: } B(r) = \frac{\mu_0}{4\pi r^3} \left[ 3(\mathbf{m} \cdot \hat{r})\hat{r} - \mathbf{m} \right] + \frac{2\mu_0}{3} \mathbf{m}\delta(r). \]

9. Relate the surface current density on a surface to the surrounding magnetic field.
   \[ \text{SOLUTION: } B_{ab} - B_{ba} = \mu_0 (K \times \hat{n}). \]

10. What is the focal length in water of a thin double concave acrylic glass lens the magnitude
    of whose radii of curvature is 10 cm?
    \[ \text{SOLUTION: } n_{\text{lens}} \approx 1.5, \ n_{\text{out}} \approx 1.33, \ R_1 = -10 \text{ cm and } R_2 = +10 \text{ cm. Then, } f^{-1} = \]
    \[ \frac{1.5 - 1.33}{1.33} \left( (-10 \text{ cm})^{-1} - (+10 \text{ cm})^{-1} \right) \approx (-40 \text{ cm})^{-1}. \text{ So, } f \approx -40 \text{ cm.} \]
11. There is a conducting half-infinite plane in the $x \geq 0$ region of the $xy$-plane, and another in the $z \geq 0$ region of the $yz$-plane. There is a quarter-spherical bump of radius $R$ at their intersection at the origin. There is a charge $Q$ at the point $\sqrt{2}R(1,0,1)$. Determine the required image charge configuration.

**SOLUTION:** $-Q$ at $\sqrt{2}R(-1,0,1)$ and $\sqrt{2}R(1,0,-1)$ and $Q$ at $-\sqrt{2}R(1,0,1)$. Then, $-Q/2$ at $\frac{R}{\sqrt{2}}(1,0,1)$ and $-\frac{R}{\sqrt{2}}(1,0,1)$ and $Q/2$ at $\frac{R}{\sqrt{2}}(-1,0,1)$ and $\frac{R}{\sqrt{2}}(1,0,-1)$.

12. What are Maxwell’s equations in matter?

**SOLUTION:** $\nabla \cdot \mathbf{D} = \rho_f$, $\nabla \times \mathbf{E} = -\partial \mathbf{B} / \partial t$, $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{H} = -\mathbf{J}_f + \partial \mathbf{D} / \partial t$.

13. What are the general EM field boundary conditions at interfaces between two regions?

**SOLUTION:** $D_1^\bot - D_2^\bot = \sigma_f$, $E_1^\parallel - E_2^\parallel = 0$, $B_1^\bot - B_2^\bot = 0$ and $H_1^\parallel - H_2^\parallel = K_f \times \mathbf{n}$.

14. What is the impedance of a capacitor and inductor connected in parallel?

**SOLUTION:** $Z^{-1} = Z_C^{-1} + Z_L^{-1} = i\omega C + \frac{1}{i\omega L} = \frac{1}{i\omega L} (1 - \omega^2 LC)$ and so $Z = \frac{i\omega L}{1 - \omega^2 LC}$.

15. A very long solenoid of radius $R_1$ has $n_1$ coils per unit length. Inside it there is a small short solenoid with radius $R_2$ and $n_2$ coils per unit length and length $L$. A current $I$ flows in the small solenoid. Calculate the magnetic flux through the large solenoid.

**SOLUTION:** Run the current through the large solenoid instead. The field inside is constant: $B = \mu_0 n_1 I$. The flux through the small solenoid is $\Phi_2 = \mu_0 n_1 I \pi R_2^2 (n_2 L) = (\mu_0 n_1 \pi R_2^2) LI$. Hence, the mutual inductance is $M_{21} = \mu_0 n_1 n_2 \pi R_2^2 L$, and so $\Phi_1 = M_{12} I = M_{21} I = \mu_0 n_1 n_2 \pi R_2^2 LI$.

16. What is the angular momentum density in EM fields?

**SOLUTION:** $\ell = \mu_0 \varepsilon_0 \mathbf{r} \times \mathbf{S}$.

17. Write down the Fresnel equations for incoming polarization parallel to the plane of incidence.

**SOLUTION:** $r_\parallel = \frac{\alpha - \beta}{\alpha + \beta}$ and $t_\parallel = \frac{2}{\alpha + \beta}$, where $\alpha = \cos \theta_F / \cos \theta_I$ and $\beta = \mu_1 n_2 / \mu_2 n_1$. The reflection and transmission coefficients are $R = r^2$ and $T = t^2$.

18. Write down the Fresnel equations for incoming polarization perpendicular to the plane of incidence.

**SOLUTION:** $r_\perp = \frac{1 - i\alpha \beta}{1 + i\alpha \beta}$ and $t_\perp = \frac{2}{1 + i\alpha \beta}$, where $\alpha = \cos \theta_F / \cos \theta_I$ and $\beta = \mu_1 n_2 / \mu_2 n_1$. The reflection and transmission coefficients are $R = r^2$ and $T = t^2$.

19. A plane EM wave is incident on a perfectly reflecting plane mirror of area $A$ and mass $m$. How strong must the electric field amplitude, $E_0$, be in order to allow the mirror to hover just above the earth’s surface?

**SOLUTION:** The intensity is $I = \frac{1}{2} \varepsilon_0 E_0^2 c$ and the pressure on the mirror is $P = 2I/c$. Thus, the radiation force exerted on it is $F = PA = \varepsilon_0 E_0^2 A$ and must cancel $mg$. Thus, $E_0 = \sqrt{mg / \varepsilon_0 A}$.

20. What is the average radiated power for electric dipole radiation?

**SOLUTION:** $\langle P \rangle = \mu_0 p_0^2 \omega^4 / 12 \pi c$, where $p_0$ is the dipole moment amplitude.

21. Write down the general electromagnetic field strength tensor.

**SOLUTION:** $F^{\ell i} = E_i / \nu$, $F^{12} = B_z$, $F^{13} = -B_y$ and $F^{23} = B_x$. The diagonals are zero and the lower-triangular entries are negatives of the upper-triangular entries.
1. What is the fundamental assumption of thermodynamics and to which ensemble does it pertain?

**SOLUTION:** It pertains to the microcanonical ensemble, which is completely isolated from the environment. The assumption is that all microstates are equally likely to be occupied.

2. What is the statistical definition of temperature? What is the kinematic “definition” of temperature?

**SOLUTION:** Statistical: \( \frac{1}{\tau} = (\partial \sigma / \partial U)_{N,V} \). Kinematic: measure of energy per particle (made slightly more precise via the equipartition theorem).

3. What is the equipartition theorem?

**SOLUTION:** Each quadratic degree of freedom contributes \( \frac{\tau}{2} \) worth of internal energy.

4. What is the Stirling approximation?

**SOLUTION:** \( \log N! \approx N \log N - N \).

5. What is the multiplicity for a two-state paramagnet in terms of the total number of spins, \( N \), and the spin excess, \( s = \frac{1}{2}(N_\uparrow - N_\downarrow) \)? Express it exactly, as well as in the large \( N \) limit.

**SOLUTION:** \( g(N, s) = \frac{N!}{(\frac{N}{2} + s)! (\frac{N}{2} - s)!} \xrightarrow{N \to \infty} g(N, 0) e^{-2s^2/N} \).

6. What is the partition function for \( N \) identical free ideal gas particles in \( d \) dimensions? [The surface area of \( S^{d-1} \) is \( 2\pi^{d/2} / \Gamma(d/2) \) and \( \Gamma(z) \equiv \int_0^\infty t^{z-1} e^{-t} \, dt \), though this is not necessary to do this problem.]

**SOLUTION:** \( Z_1 = \int_0^\infty dn_1 \cdots dn_d \exp \left[ -\hbar^2 \pi^2 (n_1^2 + \cdots + n_d^2) / 2mL^2 \tau \right] \). Define \( \alpha = \hbar \pi / \sqrt{2m\tau} L \). Then, \( Z_1 = \left[ \int_0^\infty dne^{-\alpha^2 n^2} \right]^d = (\pi / 4\alpha^2)^{d/2} = n_Q V_d \), where \( n_Q = (m\tau / 2\pi \hbar^2)^{d/2} \). Then, \( Z_n = Z_1^n / N! \).

7. What is the thermodynamic identity?

**SOLUTION:** \( dU = \tau d\sigma - pdV + \mu dN \).

8. Define the canonical ensemble.

**SOLUTION:** Allowed to exchange energy with the environment.

9. Write down the grand canonical partition function.

**SOLUTION:** \( Z = \sum_{s,N} e^{-(\tau \sigma - \mu N) / \tau} \).
10. What is the Sackur-Tetrode result?
   **SOLUTION:** \( \sigma = N \left[ \log(n_Q/n) + \frac{5}{2} \right], \) for identical monoatomic free ideal gas particles.

11. Describe our mnemonic for the thermodynamic potentials.
   **SOLUTION:** However Very Good Students Fail Pretty Easy Tests. This gives the changes in the thermodynamic potentials. For example, \( \Delta G = V \Delta P - S \Delta T \). It can also tell you the appropriate Legendre transformations among the potentials. For example, \( G = U - TS + PV \) (in going from \( U \) to \( G \), one must go from \( T \) to \( S \) (parallel to the arrow, which in this case implies a minus sign) and from \( P \) to \( V \) (anti-parallel to the arrow, which in this case implies a plus sign). Notice the slight sign difference between how one gets the \( \Delta \)'s and how one gets the Legendre transformations.

12. What is the definition of two substances being in diffusive equilibrium?
   **SOLUTION:** \( \mu_1 = \mu_2. \)

13. What is the chemical potential for a free ideal gas?
   **SOLUTION:** \( \mu = \tau \log(n/n_Q). \)

14. Express the Fermi energy in 3D in terms of relevant quantities.
   **SOLUTION:** \( \varepsilon_F = \left( \frac{\hbar^2}{2m} \right) \left( \frac{3 \pi^2 n}{2} \right)^{2/3}, \) where \( n = N/V. \)

15. What is the heat capacity for an electron gas well below the Fermi temperature?
   **SOLUTION:** \( C_e = \frac{1}{3} \pi^2 D(\varepsilon_F) \tau = \frac{1}{2} \pi^2 N \tau / \varepsilon_F. \)

16. What is the Einstein condensation temperature at arbitrary dimension \( d \geq 2? \)
   **SOLUTION:** \( \tau_E = \frac{2 \pi \hbar^2}{m(n/\zeta(d/2))^{2/d}}, \) where \( n = N/V. \)

17. What is the Clausius-Clapeyron relation and what does it do for you?
   **SOLUTION:** \( \frac{dp}{dT} = \frac{s_1 - s_2}{v_1 - v_2}, \) where \( s \) and \( v \) are entropy and volume per mole. We can also write this as \( \frac{L}{\tau \Delta T}, \) where \( L \) is the latent heat of vaporization per mole. This allows us to trace the coexistence curve between phases 1 and 2 on a pressure-temperature diagram if we know one point on that curve.

18. Write down the Maxwell speed distribution.
   **SOLUTION:** \( P(v) = 4\pi \left( \frac{m}{2\pi \tau} \right)^{3/2} v^2 e^{-mv^2/2\tau}. \)

19. What is the heat flow equation?
   **SOLUTION:** \( \dot{Q} = -KA(dT/dx), \) where \( K \) is the thermal conductivity, \( A \) is the cross-sectional area through which the heat is flowing, and \( x \) is the direction of heat flow.

20. Describe how a heat shield works.
   **SOLUTION:** Suppose one plane sits at temperature \( T_h \) and another parallel one sits at \( T_l < T_h. \) Place a heat shield in between, at \( T_m. \) The leftward flux hitting \( T_l \) is \( J_m - J_l \) and the leftward flux hitting the heat shield is \( J_h - J_m. \) At equilibrium, these must be equal and thus \( J_m = (J_h + J_l)/2. \) Hence, the flux hitting \( T_l \) is \( J_m - J_l = \frac{1}{2}(J_h - J_l), \) which is half of what it would have been without the heat shield.

21. What is the approximate value of room temperature in eV?
   **SOLUTION:** \( \sim \frac{1}{40} \) eV.

22. Which thermodynamic potential is most useful for simultaneously isothermal and isochoric processes?
   **SOLUTION:** Helmholtz free energy since \( \Delta F = -S \Delta T - p \Delta V = 0 \) in this case.
Quantum Mechanics Quiz Solutions

1. What is the numerical value and expression in terms of fundamental constants of the fine
structure constant?
   SOLUTION: \( \alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c} \approx 1/137. \)

2. Write \( L_{\pm} |\ell, m\rangle \) in the \( |\ell, m\rangle \) basis.
   SOLUTION: \( L_{\pm} |\ell, m\rangle = \hbar \sqrt{(\ell + 1) - m(m \pm 1)} |\ell, m \pm 1\rangle \).

3. Define the gyromagnetic ratio.
   SOLUTION: \( \mu = S \), where \( \mu \) is the magnetic dipole moment and \( S \) is the spin operator.

4. Write down Schrödinger’s equation. If the potential is time-independent, what is the most
   general form of the wavefunction.
   SOLUTION: \( i\hbar \dot{\psi} = H\psi \), where \( H = -\frac{\hbar^2}{2m}\nabla^2 + V \). If \( V \) is \( t \)-independent, then \( \psi(x, t) = \psi(x)e^{-iEt/\hbar} \) where \( H\psi = E\psi \).

5. For a harmonic oscillator, \( (a^+_+)^{-1} = M\rho \). What is the matrix \( M \) and its inverse?
   SOLUTION: \( M = (2\hbar \mu \omega)^{-1/2}(\mu \omega^{-1} - i) \) and \( M^{-1} = \sqrt{\hbar/2m\omega}\left(\mu \omega^{-1} - i\mu \omega\right) \).

6. Write down the generalized uncertainty principle for two operators, \( A \) and \( B \).
   SOLUTION: \( \sigma_A\sigma_B = \frac{1}{2} \langle[A, B]\rangle \).

7. Write down the time-evolution of the expectation value of an operator.
   SOLUTION: \( \frac{d}{dt} \langle Q \rangle = \frac{i}{\hbar} \langle[H, Q]\rangle + \langle \frac{\partial Q}{\partial t} \rangle \).

8. What is the generic form of the radial equation for a spherically symmetric potential?
   SOLUTION: \( -\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + V_{\text{eff}}(r) \) \( u(r) = Eu(r) \), where \( V_{\text{eff}} = V + \frac{\hbar^2(\ell + 1)}{2mr^2} \) and the actual radial wavefunction is \( R = u/r \).

9. Write down the general form of the hydrogen wavefunctions. Describe the radial component
   in as much detail as you can.
   SOLUTION: \( \psi_{n\ell m} = R_{n\ell}(r)Y_{\ell m}(\Omega) \), where \( n = 1, 2, \ldots \) and \( \ell = 0, 1, \ldots, n - 1 \) and \( m = -\ell, \ldots, \ell \). In general, \( R_{n\ell} \sim e^{-r/a}x^\ell f_{n-\ell-1}(x) \), where \( x = r/a \) and \( a \) is the appropriate Bohr radius, and \( f_{n-\ell-1}(x) \) is some degree \( n - \ell - 1 \) polynomial in \( x \).

10. What is the Hamiltonian for a spin in a magnetic field?
    SOLUTION: \( H = -\mu \cdot B = -\gamma S \cdot B \).
11. What are the possible total angular momentum quantum numbers (and their degeneracies) for three electrons two of which are in \( s \) states and one of which is in a \( p \) state?

**SOLUTION:** The possible total spins are \( S = \frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = (0 \oplus 1) \otimes \frac{1}{2} \otimes \frac{1}{2} = 2(\frac{1}{2}) \oplus \frac{3}{2} \). The possible total orbital angular momenta are \( L = 0 \otimes 0 \otimes 1 = 1 \). Thus, the possible values of total angular momentum are \( J = [2(\frac{1}{2}) \oplus \frac{3}{2}] \otimes 1 = 3(\frac{1}{2}) \oplus 3(\frac{3}{2}) \oplus \frac{5}{2} \).

12. The \( \ell = 1 \) spherical harmonics are \( Y_{1}^{\pm 1} = \mp \sqrt{3/8\pi} \sin \theta e^{\pm i\phi} \) and \( Y_{1}^{0} = \sqrt{3/4\pi} \cos \theta \). Can you think of a quick argument as to why \( \langle \ell' m'|x|\ell m \rangle = 0 \) unless \( \Delta m = \pm 1 \) and \( \Delta \ell = 0, \pm 1 \)?

**SOLUTION:** Wigner-Eckart: \( x = \sqrt{2\pi/3} r(Y_{1}^{1} - Y_{1}^{-1}). \) When this acts on \( |\ell m\rangle \), by addition of angular momentum, the possible final values are \( \ell' = \ell \otimes 1 = \ell, \ell + 1 \), and thus \( \Delta \ell = 0, \pm 1 \). Likewise, \( m \) must change by \( \pm 1 \).

13. Write the electron configuration of Titanium (\( Z = 22 \)) and use Hund’s rules to determine the ground state.

**SOLUTION:** \([\text{Ti}] = 1s^{2}2s^{2}2p^{6}3s^{2}3p^{6}4s^{2}3d^{2}. \) The only unfilled subshell is the last one, which contains two electrons (\( s = 1/2 \)) in \( \ell = 2 \) states. The possible spin states are \( S = 0 \) (singlet, antisymmetric) and \( S = 1 \) (triplet, symmetric). The possible \( L \)'s are \( L = 2 \otimes 2 = 0 \oplus 1 \oplus 2 \otimes 3 \oplus 4 \), with parity \((-1)^{L} \). Hence, the possible states are \( 1S_{0}, 1D_{2}, 1G_{4}, 3P_{0,1,2}, \) and \( 3F_{2,3,4}. \) First, we must maximize \( S \), which picks out the last two sets. Then, we must maximize \( L \), which picks out the last set. Since the last subshell is \( \leq \) half-full, we must minimize \( J \), so the ground state is \( 3F_{2} \).


**SOLUTION:** The two-electron wavefunction for parahelium is spatially symmetric and for orthohelium is spatially anti-symmetric.

15. What is the first order time-independent perturbation energy correction?

**SOLUTION:** \( E_{1}^{(1)} = H'_{mn}, \) where \( H'_{mn} = \langle m|H'|n \rangle \) and \( H' \) is the small perturbation.

16. What is the first order time-independent perturbation wavefunction correction?

**SOLUTION:** \( \psi_{n}^{(1)} = \sum_{m \neq n} (H'_{mn}/\Delta_{nm})\psi_{m}^{(0)}, \) where \( \Delta_{nm} = E_{n}^{(0)} - E_{m}^{(0)} \).

17. What is the second order time-independent perturbation energy correction?

**SOLUTION:** \( E_{2}^{(2)} = \sum_{m \neq n} |H'_{nm}|^{2}/\Delta_{nm}. \)

18. Write \( a_{\pm} |n \rangle \) in the \( |n \rangle \) basis.

**SOLUTION:** \( a_{\pm} |n \rangle = (n + \frac{1}{2} \pm \frac{1}{2})^{1/2} |n \pm 1 \rangle. \)

19. Consider the two-dimensional rectilinear SHO, which is just a sum of SHO Hamiltonians in the \( x \) and \( y \) directions, with the same frequency. Add a perturbation \( H' = \frac{1}{m\omega} E_{xy} \). Calculate the perturbation matrix in subspace with unperturbed energy \( 2h\omega. \) How is this energy perturbed?

**SOLUTION:** \( E = \hbar \omega (n_{x} + n_{y} + 1) \) and so the two states, \( |1 \rangle \equiv |01 \rangle \) and \( |2 \rangle \equiv |10 \rangle \), both have energy \( 2h\omega. \) Write \( x = \sqrt{\hbar/2m\omega}(a_{x}^{\dagger} + a_{x}) \) and similarly for \( y \). Thus, define \( \epsilon = \frac{\hbar}{2m\omega} \frac{1}{50} m\omega^{2} = \hbar/100. \) Then \( H' = \epsilon(a_{x}^{\dagger} + a_{x})(a_{y}^{\dagger} + a_{y}) \), and the perturbation matrix, defined as \( M_{ij} = \langle i|H'|j \rangle \), is given by \( M = \epsilon (n_{0}^{1} 0) \). When, diagonalized, this matrix reads \( M_{\text{diag}} = \epsilon (n_{0}^{1} 0) \) and thus the energy is perturbed to \( E_{\pm} = 2\hbar \omega \pm \epsilon = (2 \pm \frac{1}{100})\hbar \omega. \)

20. What is the variational principle?

**SOLUTION:** \( E_{gs} \leq \langle \psi |H|\psi \rangle \) for any normalized wavefunction, \( |\psi \rangle \).
21. Write down the differential equation determining the expansion coefficients in time-dependent perturbation theory.

**SOLUTION:** \( \dot{c}_m = -\frac{i}{\hbar} \sum_n H'_{mn} e^{i\omega_m t} c_n \), where \( \omega_{mn} = (E_m - E_n)/\hbar \).

22. What is Fermi’s golden rule?

**SOLUTION:** \( w = \frac{2\pi}{\hbar} |\langle f | V | i \rangle|^2 \rho_f(E_i) \), where \( \rho_f(E_i) \) is the density of final states evaluated at the initial energy.

23. What is the general expression for the differential scattering cross section in the first Born approximation?

**SOLUTION:** \( \frac{d\sigma}{d\Omega} = |f(\Omega)|^2 = \frac{2\pi m^2}{\hbar^2} |V(q)|^2 \).

24. What is the scattering cross section for s-wave scattering?

**SOLUTION:** \( \sigma = \frac{4\pi}{\hbar^2} \sin^2 \delta_0 \), where \( k = \sqrt{2mE}/\hbar \) and the outgoing u-wavefunction is \( u(r) \sim \sin(kr + \delta_0) \).

25. Write down a complete set of commuting observables for hydrogen including fine structure effects.

**SOLUTION:** \( H, J^2 \) and \( J_z \).

26. Write down a complete set of commuting observables for hydrogen including the strong-field Zeeman effect.

**SOLUTION:** \( H, L^2, L_z, S_z \).

27. What is wrong with the following argument? A 1D finite rectangular well always has at least one bound state regardless of how narrow and shallow the well is. This violates the uncertainty principle since it implies that a particle with arbitrarily small energy, hence momentum, can be confined to an arbitrarily small space.

**SOLUTION:** In fact the particle is not confined to the well, it can penetrate the classically forbidden region. In fact, the smaller the energy of the particle, which corresponds to a shallower well, the more deeply it penetrates the classically forbidden region, thus preserving the uncertainty principle.

28. In the case of the potential downward step \( (V = -V_0 \theta(x)) \), the expressions for reflection and transmission can be written in terms of \( E \) and \( E - V_0 \) only and are independent of \( \hbar \). Resolve the following ostensible paradox: the classical limit is generically reached by sending \( \hbar \) to zero. But \( T \) and \( R \) are independent of \( \hbar \) and thus are unchanged by this limit. Hence, there is partial reflection even in the classical limit!

**SOLUTION:** There is no consistent classical limit for this idealized potential. If it is indeed infinitely steep at the step, then it is impossible for the de Broglie wavelength of the incoming particle to be smaller than the changes in the potential, which is really the definition of the classical limit here. Indeed, in the \( E \to \infty \) limit, the reflection coefficient does vanish: \( R = \left( \frac{E}{V_0} \right)^2 \left( 1 - \sqrt{1 - \frac{V_0}{E}} \right)^4 \).

29. Let \( V_j = \infty \) for \( x < 0 \), \( 0 \) for \( 0 \leq x < L_j \) and \( L_j + w \leq x \), and \( V_0 > 0 \) for \( L_j \leq x < L_j + w \), for \( j = 1, 2 \) and \( L_1 > L_2 \). Initially, a particle with \( E < V_0 \) is localized to the left of the barrier. At some later time, which case will have a greater probability of the particle still being in the well?

**SOLUTION:** Case 1. Since the energies are the same, as are the momenta. But, well 1 is larger than well 2, so the particle in well 1 will hit the barrier less frequently than does the particle in well 2 and thus will have a lower penetration rate.
30. What is wrong with the following argument? For the previous problem, place the particle in an infinite square well wavefunction for $0 \leq x < L_j$ and let $\psi = 0$ everywhere else. This satisfies the S.E. in the well and trivially everywhere else. Therefore, both particles are in stationary states and the particles will never leave the wells.

**SOLUTION:** This wavefunction does not satisfy the boundary condition that $d\psi/dx$ be continuous at $x = L_j$. 